Inverse Scattering in One-Dimensional Nonconservative Media

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Dedicated to M.G. Krein, one of the founding fathers of inverse scattering theory.

The inverse scattering problem arising in wave propagation in one-dimensional nonconservative media is analyzed. This is done in the frequency domain by considering the Schrödinger equation with the potential $ikP(x) + Q(x)$, where $k^2$ is the energy and $P(x)$ and $Q(x)$ are real integrable functions. Using a pair of uncoupled Marchenko integral equations, $P(x)$ and $Q(x)$ are recovered from an appropriate set of scattering data including bound-state information. Some illustrative examples are provided.

0. INTRODUCTION

The wave propagation in a one-dimensional medium, where energy absorption or generation may occur, can be described in the frequency domain by the generalized Schrödinger equation

$$\psi^{\prime\prime}(k, x) + k^2 \psi(k, x) = [ikP(x) + Q(x)] \psi(k, x), \quad x \in \mathbb{R}, \quad (0.1)$$

where $\mathbb{R}$ is the real line, the prime denotes the derivative with respect to the spatial coordinate $x$, $k$ is the wavenumber, $k^2$ is the energy, $P(x)$ represents the energy absorption or generation, and $Q(x)$ represents the restoring force density. By changing the sign of $P(x)$ in (0.1) we obtain the associated equation

$$\psi^{\prime\prime}(k, x) + k^2 \psi(k, x) = [-ikP(x) + Q(x)] \psi(k, x), \quad x \in \mathbb{R}, \quad (0.2)$$

whose scattering data are to be used along with the scattering data from (0.1) in order to recover $P(x)$ and $Q(x)$.

Let $L^p_q(I)$ denote the measurable functions $f(x)$ such that $\int_I dx (1 + |x|)^q |f(x)|^p$ is finite. Note that we have $L^p(I) = L^p_0(I)$. We will assume that $Q(x)$ is real valued and belongs to $L^1_1(\mathbb{R})$ and that $P(x)$ is real valued and satisfies $P \in L^1(\mathbb{R})$. We will use $||f||_p$ to denote the norm on $L^p(\mathbb{R})$ and write $||f||_{1,q}$ for $\int_{-\infty}^{\infty} dx (1 + |x|)^q |f(x)|$. We will later impose further restrictions on $P(x)$ and $Q(x)$.

The scattering solutions of (0.1) and (0.2) comprise those behaving like $e^{ikx}$ or $e^{-ikx}$ as $x \to \pm \infty$, and such solutions occur when $k^2 > 0$. Among the scattering solutions are the
Jost solution from the left $f_1^\pm (k, x)$ and the Jost solution from the right $f_r^\pm (k, x)$ satisfying the boundary conditions

$$
 f_1^\pm (k, x) = \begin{cases} 
 e^{ikx} + o(1), & x \to +\infty, \\
 \frac{1}{T^\pm (k)} e^{ikx} + \frac{L^\pm (k)}{T^\pm (k)} e^{-ikx} + o(1), & x \to -\infty,
\end{cases} \quad (0.3)
$$

$$
 f_r^\pm (k, x) = \begin{cases} 
 \frac{1}{T^\pm (k)} e^{-ikx} + \frac{R^\pm (k)}{T^\pm (k)} e^{ikx} + o(1), & x \to +\infty, \\
 e^{-ikx} + o(1), & x \to -\infty,
\end{cases} \quad (0.4)
$$

where $T^\pm (k)$ is the transmission coefficient and $R^\pm (k)$ and $L^\pm (k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrices $S^+ (k)$ and $S^- (k)$ associated with (0.1) and (0.2), respectively, are given by

$$
 S^\pm (k) = \begin{bmatrix} T^\pm (k) & R^\pm (k) \\ L^\pm (k) & T^\pm (k) \end{bmatrix}.
$$

Let $[F; G] = FG' - F'G$ denote the Wronskian. The scattering coefficients can be expressed in terms of Wronskians of the Jost solutions of (0.1) and (0.2) as

$$
 [f_1^\pm (k, x); f_r^\pm (k, x)] = \frac{-2ik}{T^\pm (k)}, \quad k \in \mathbb{C}^+ ,
$$

$$
 [f_1^\pm (k, x); f_r^\mp (-k, x)] = \frac{2ik L^\pm (k)}{T^\pm (k)} = -\frac{2ik R^\mp (-k)}{T^\pm (-k)}, \quad k \in \mathbb{R},
$$

$$
 [f_r^\pm (k, x); f_1^\mp (-k, x)] = -\frac{2ik R^\pm (k)}{T^\pm (k)} = \frac{2ik L^\mp (-k)}{T^\pm (-k)}, \quad k \in \mathbb{R}.
$$

We have [JJ76a,AKV97]

$$
 S^\pm (-k) = S^\mp (k), \quad k \in \mathbb{R},
$$

$$
 S^\pm (k) S^\mp (-k)^t = I, \quad k \in \mathbb{R},
$$

where $I$ is the $2 \times 2$ unit matrix, the superscript $t$ denotes the matrix transpose, and the overline denotes complex conjugation. From (0.7) we get

$$
 L^\pm (k) T^\mp (-k) + T^\pm (k) R^\mp (-k) = 0, \quad k \in \mathbb{R},
$$

$$
 T^\pm (k) T^\mp (-k) = 1 - R^\pm (k) R^\mp (-k), \quad k \in \mathbb{R}.
$$

The bound-state solutions of (0.1) and (0.2) are those nontrivial solutions belonging to $L^2 (\mathbb{R})$. Such solutions correspond to the values of $k \in \mathbb{C}^+$ at which the Jost solutions