OPERATOR INEQUALITIES AND CONSTRUCTION OF KREIN SPACES

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Let $\mathcal{H}$ be a Hilbert space. A continuous positive operator $T$ on $\mathcal{H}$ uniquely determines a Hilbert space $\mathcal{G}$ which is continuously imbedded in $\mathcal{H}$ and for which $T = E_\mathcal{G} E_\mathcal{G}^*$ with the canonical imbedding $E_\mathcal{G}$. A Krein space version of this result, however, is not valid in general. This paper provides a necessary and sufficient condition for that a continuous selfadjoint operator $T$ uniquely determines a Krein space $(\mathcal{K}, J_\mathcal{K})$ which is continuously imbedded in $\mathcal{H}$ and for which $T = E_\mathcal{K} J_\mathcal{K} E_\mathcal{K}^*$ with the canonical imbedding $E_\mathcal{K}$.

1 INTRODUCTION

The complementation theorem in Hilbert spaces, discovered by L. de Branges and J. Rovnyak [4], has many applications to operator theory as seen in the monograph of L. de Branges [3]. A corresponding version of the complementation theorem for Krein spaces (i.e. the indefinite case) was also developed by L. de Branges [2], but it has many differences from the definite case. For instance, a uniqueness theorem, which is obvious in the definite case, is no longer valid in the indefinite case.

Fix a Hilbert space $\mathcal{H}$. A Hilbert space $\mathcal{G}$ is said to be continuously imbedded in $\mathcal{H}$ if $\mathcal{G}$ is contained in $\mathcal{H}$ as a linear subspace and the canonical imbedding $E_\mathcal{G}$ is continuous. The canonical imbedding $E_\mathcal{G}$ defines a continuous positive (semi-definite) operator $T$ on $\mathcal{H}$ by $T \equiv E_\mathcal{G} E_\mathcal{G}^*$. Conversely, a continuous positive operator $T$ uniquely determines a Hilbert space $\mathcal{G}$ which is continuously imbedded in $\mathcal{H}$ and for which $T = E_\mathcal{G} E_\mathcal{G}^*$. Therefore there is a one-to-one correspondence between the family of Hilbert spaces which are continuously imbedded in $\mathcal{H}$ and the class of continuous positive operators on $\mathcal{H}$.

When a Krein space $(\mathcal{K}, J_\mathcal{K})$ with a canonical involution $J_\mathcal{K}$ is continuously imbed-
ded in $\mathcal{H}$, the canonical imbedding $E_{\mathcal{K}}$ defines a continuous selfadjoint operator $T$ on $\mathcal{H}$ by $T \equiv E_{\mathcal{K}}J_{\mathcal{K}}E_{\mathcal{K}}^*$. Our problem in this paper is to find a condition for that a continuous selfadjoint operator $T$ uniquely determines a Krein space $(\mathcal{K}, J_{\mathcal{K}})$ which is continuously imbedded in $\mathcal{H}$ and for which $T = E_{\mathcal{K}}J_{\mathcal{K}}E_{\mathcal{K}}^*$.

After having completed this paper, we were informed that M. A. Dritschel [6] and M. A. Dritschel and J. Rovnyak [7],[8] treated the same uniqueness problem from another point of view. Their approach, however, is completely different from ours.

2 PRELIMINARY

A Krein space $(\mathcal{K}, J_{\mathcal{K}})$ is a complex Hilbert space with a selfadjoint involution $J_{\mathcal{K}}$, called the canonical involution, that is, $J_{\mathcal{K}} = J_{\mathcal{K}}^*$ and $J_{\mathcal{K}}^2 = I$. When the Hilbert space inner product on $\mathcal{K}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$, the canonical involution $J_{\mathcal{K}}$ induces an indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ on $\mathcal{K}$ by

$$\langle x, y \rangle_{\mathcal{K}} \equiv \langle J_{\mathcal{K}}x, y \rangle_{\mathcal{K}} \quad \text{for all } x, y \in \mathcal{K}. \quad (1)$$

The canonical involution $J_{\mathcal{K}}$, hence the Hilbert space inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$, can not be described in terms of the indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ only. But the strong topology, induced by a Hilbert space inner product $\langle \cdot, \cdot \rangle$, for which (1) is valid with $\langle \cdot, \cdot \rangle$ and a selfadjoint involution $J$ in place of $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and $J_{\mathcal{K}}$ respectively, is uniquely determined by the indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$.

Let $(\mathcal{K}, J_{\mathcal{K}})$ be a Krein space. Consider two orthogonal projections $P_+ \equiv \frac{(J_{\mathcal{K}} J_{\mathcal{K}} + J_{\mathcal{K}})}{2}$, $P_- \equiv \frac{(J_{\mathcal{K}} - J_{\mathcal{K}})}{2}$ and their range subspaces $\mathcal{K}_+ \equiv \text{ran}(P_+)$, $\mathcal{K}_- \equiv \text{ran}(P_-)$. Besides usual orthogonality and the decomposition $\mathcal{K} = \mathcal{K}_+ + \mathcal{K}_-$ (algebraic direct sum), $\mathcal{K}_+$ and $\mathcal{K}_-$ have orthogonality in terms of the indefinite inner product. The decomposition is called a canonical decomposition of $\mathcal{K}$ and denote by $\mathcal{K} = \mathcal{K}_+ [+] \mathcal{K}_-$ to express orthogonality in terms of $\langle \cdot, \cdot \rangle_{\mathcal{K}}$.

A continuous linear operator $V$ on a Krein space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ is said to be $J$-isometric if

$$\langle Vx,Vy \rangle_{\mathcal{K}} = \langle x,y \rangle_{\mathcal{K}} \quad \text{for all } x,y \in \mathcal{K},$$

and a $J$-isometric surjection is said to be $J$-unitary.