THE CLOSURE OF A SIMILARITY ORBIT IS ALWAYS ARCWISE CONNECTED

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If T, A are Hilbert space operators such that A is the norm-limit of operators similar to T, then there exists a continuous function γ from [0,1] into the algebra of operators such that γ(t) is a limit of similarities of T for all t in the interval [0,1], γ(0) = T and γ(1) = A.

INTRODUCTION

Let H be a complex, separable, infinite dimensional Hilbert space, and let L(H) denote the Banach algebra of all (bounded linear) operators acting on H. The similarity orbit of T ∈ L(H) is the set

\[ S(T) = \{WTW^{-1}: W ∈ L(H), W \text{ is invertible} \}. \]

The analysis of the (norm) closure \( S(T)^{-} \) of the similarity orbit of an operator T is the core of the monograph [1] [4]. The analogous problem for the unitary orbit

\[ U(T)^{-} = \{UTU^*: U ∈ L(H), U \text{ is unitary} \} \]

was completely solved by D. W. Hadwin in [3], with the help of Voiculescu's theorem [10]. In particular, Hadwin proved that \( U(T)^{-} \) is always arcwise connected; furthermore, there exists \( T_{0} \) in \( U(T)^{-} \) such that for each A in \( U(T)^{-} \) there is a continuous function \( γ:[0,1] \rightarrow U(T)^{-} \) such that

\[ γ(0) = T_{0}, \ γ([0,1]) \subseteq U(T_{0}), \text{ and } γ(1) = A \]

(see [3,Theorem 3.9]; \( U(T_{0})^{-} = U(T)^{-} = U(A)^{-} \) for all A in \( U(T)^{-} \)).

What is the answer to the corresponding problem for \( S(T)^{-} \)? First of all, observe that this problem admits several answers, not just one. To begin with, if A ∈ \( S(T)^{-} \) then the most we can say is that \( S(A)^{-} \subseteq S(T)^{-} \); but, in general, we cannot 1) This research was partially supported by a Grant of the National Science Foundation.
expect to have \( S(A)^- = S(T)^- \). (Compare with the case of \( U(T)^- \).) For instance, the spectrum of \( A \), \( \sigma(A) \), can be strictly larger than \( \sigma(T) \), or \( T \) can be a nilpotent of order \( n \), and \( A \) a nilpotent of order \( n-1 \), etc. (see [1, Theorem 9.2]). If \( A \in S(T)^- \) and \( T \in S(A)^- \) (equivalently, \( S(T)^- = S(A)^- \)), we say that \( T \) and \( A \) are \textit{asymptotically similar}. (In symbols: \( T \# A \).)

Is \( S(T)^- \) an arcwise connected set for all \( T \in L(H) \)? The answer is YES, and the proof is relatively simple by using the main results of [1].

Thus, if \( A \in S(T)^- \), there exists a continuous function \( \gamma: [0,1] \to S(T)^- \) such that \( \gamma(0) = T \) and \( \gamma(1) = A \). Is it possible to find \( \gamma \) so that \( \gamma([0,1]) \subseteq S(T) \)? The author conjectures that the answer is NO, even if \( T \# A \). A candidate for a possible counterexample is the following pair:

Let \( \{e_n\}_{n=1}^\infty \) be an orthonormal basis of \( H \), and let \( N \) be the compact normal operator defined by \( Ne_1 = Ne_2 = 0, Ne_n = (1/n)e_n \) for \( n \geq 3 \) (\( N = \text{diag}[0,0,1/3,1/4,1/5,...] \)). If \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), then \( A \# N \) [4, Chapter 5]. Is there any continuous function \( \gamma: [0,1] \to S(N) \cup \{A\} \) such that \( \gamma(0) = N, \gamma(1) = A \), and \( \gamma([0,1]) \subseteq S(N) \)?

Suppose \( \tau \) is a spectral function, and \( \tau \) is \textit{discontinuous} at \( A \in L(H) \) (e.g., \( \tau \) is the spectrum \( \sigma(.) \), or the essential spectrum \( \sigma_e(.) \), or the spectral radius \( \text{sp}(.), \) etc.). Is it possible to construct a continuous arc \( \gamma: [0,1] \to L(H) \) such that \( \tau \circ \gamma \) is continuous on \([0,1] \), but discontinuous at \( 1 \)? This problem was analyzed in [1, Section 14.9]. Among other positive results, we have the following: there exists a \( C^1 \)-arc \( \gamma \), real-analytic for \( 0 \leq t < 1 \), such that \( \gamma(t) \) is quasinilpotent for \( 0 \leq t < 1 \), but \( \gamma(1) \) has positive spectral radius.

As observed in the above reference, an affirmative answer to the following conjecture will also affirmatively answer most of the problems in this area:

CONJECTURE ([1, Conjecture 14.10]). If \( T \in L(H) \) and \( S(T)^- \) is maximal (with respect to set inclusion), then given \( A \)