ASYMPTOTIC BEHAVIOUR OF LINEAR EVOLUTIONARY INTEGRAL EQUATIONS

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The asymptotic behaviour of bounded solutions of evolutionary integral equations in a Banach space $X$

$$
\dot{u}(t) = \int_{0}^{\infty} A_{0}(\tau) \dot{u}(t - \tau) \, d\tau + \int_{0}^{\infty} dA_{1}(\tau) u(t - \tau) + f(t), \quad t \in \mathbb{R},
$$
on the real line and of

$$
\dot{v}(t) = \left( \int_{0}^{t} A(t - \tau)v(\tau) \, d\tau \right) + g(t), \quad t \in \mathbb{R}_{+},
$$
on the half-line are studied. Assuming that the inhomogeneity $f$ (resp. $g$) belongs to a given homogeneous subspace $\mathcal{E}$ of $BUC(\mathbb{R}; X)$ (resp. $BUC(\mathbb{R}_{+}; X)$) it is shown that given bounded solutions $u$ (resp. $v$) belong also to $\mathcal{E}$ provided the spectra of these equations are countable. The results are applied to an equation of scalar type which is of importance in applications like viscoelasticity.

0 Introduction

Let $X$ be a Banach space, $A$ the generator of a $C_{0}$-semigroup, and consider the evolution equation

$$
\dot{u}(t) = Au(t) + f(t), \quad t \in \mathbb{R},
$$
where $f \in L^{\infty}(\mathbb{R}; X)$, as well as the abstract Cauchy problem

$$
\dot{v}(t) = Av(t) + g(t), \quad v(0) = v_{0}, \quad t \in \mathbb{R}_{+},
$$
where $g \in L^{\infty}(\mathbb{R}_{+}; X)$, $v_{0} \in X$, and the dot indicates the derivative with respect to time. As is well known, the unique mild solution of (0.2) is then given by

$$
v(t) = S(t)v_{0} + \int_{0}^{t} S(\tau)g(t - \tau) \, d\tau, \quad t \in \mathbb{R}_{+},
$$
where \((S(t))_{t \geq 0}\) denotes the semigroup generated by \(A\), and in case the type \(\omega_0(A)\) of \((S(t))_{t \geq 0}\) is negative, the unique mild solution of (0.1) is

\[
u(t) = \int_0^\infty S(\tau)f(t - \tau)\, d\tau, \quad t \in \mathbb{R}, \tag{0.4}\]

Such variation of parameter formulae arise also in the context of linear evolutionary integral equations which include (0.1) and (0.2) as special cases.

\[
u(t) = \int_0^\infty A_0(\tau)\dot{u}(t - \tau)\, d\tau + \int_0^\infty dA_1(\tau)u(t - \tau) + f(t), \quad t \in \mathbb{R}, \tag{0.5}\]
on the real line and

\[
u(t) = \left(\int_0^t A(t - \tau)v(\tau)\, d\tau\right) + g(t), \quad v(0) = v_0, \quad t \in \mathbb{R}_+, \tag{0.6}\]
on the half-line. In this case \((S(t))_{t \geq 0}\) denotes the so-called resolvent family, and (0.3) and (0.4) make sense if the resolvent family exists and is integrable. As a general reference for evolutionary integral equations we refer to the second author's monograph [18].

The asymptotic behaviour of the solutions of (0.5) and (0.6) in the stable case, i.e. in case the resolvent family exists and is integrable, is well-known; see Prüss [18] and Prüss and Ruess [19]. In fact, it was shown there that convolutions with operator-valued BV-functions are leaving invariant homogeneous spaces on the line and that solutions \(u\) and \(v\) are asymptotic to each other provided \(f\) and \(g\) have this property.

The study of the asymptotic behaviour of solutions of (0.5) and (0.6) in case \((S(t))_{t \geq 0}\) is not integrable or does not even exist is much more difficult and only few general results are so far known. It was begun in the paper of the second author [17]. Next we mention the paper Arendt and Prüss [4] who used vector-valued Tauberian theorems to study the asymptotic behaviour of the resolvent \((S(t))_{t \geq 0}\) in the so-called scalar case \(A(t) = \alpha(t)A\), where \(A\) denotes an unbounded closed linear operator in \(X\) and \(\alpha \in L^1_{\text{loc}}(\mathbb{R}^+)\) a scalar kernel. In [18, Sections 11, 12], the so-called non-resonant case for (0.5) is studied in detail. Non-resonant means that the spectrum \(\Lambda_0\) of (0.5) does not intersect the Carleman spectrum of the right hand side \(f\); see below for the definition of these concepts. If in addition \(\Lambda_0 = \emptyset\) and (0.6) admits a resolvent family it is shown that solutions \(u\) and \(v\) are asymptotic to each other, provided \(f\) and \(g\) are so, and certain regularity conditions are satisfied.

More recently, considerable progress concerning the asymptotic behaviour of bounded solutions of evolution equations (0.1) and (0.2) has been made; cf. Arendt and Batty [2], [3], Batty, van Neerven and Räbiger [9], [10], Chill [12], Ruess and Vu [21]. To mention one of the most prominent results in this new line of research, suppose that \(f\) is almost periodic, \(\sigma(A) \cap i\mathbb{R}\) is countable, and \(u\) is a bounded, uniformly continuous and uniformly ergodic (see Section 1 for the definition) solution of (0.1). Then \(u\) is almost periodic.

It is the purpose of this article to extend results of this type to evolutionary integral equations (0.5) and (0.6). It turns out that such extensions are valid in a fairly general setting, however, require proofs different from those for evolution equations, and which are due to enlarged generality more sophisticated. They heavily employ recent results from harmonic analysis of bounded vector-valued functions.