SOME CONTINUOUS INDEX THEOREMS

H. Schröder

In this paper we give criteria of Fredholmness and index theorems for Toeplitz operators whose symbols take values in a $W^*$-algebra factor of type $II_1$.

1. INTRODUCTION

Criteria of Fredholmness and index theorems for Toeplitz operators (on $S^1$) have been proved for the first time by Gohberg and Krein in the late fifties. In the late sixties and early seventies, with the aid of Banach algebra techniques and refined methods of algebraic topology, Coburn, Douglas, Howe, Singer, and Venugopalkrishna extended these results to Toeplitz operators with matricial symbols on $S^{2n-1}$ and on $S^1 \times S^1$. See the survey-paper [8] of Douglas for these classical discrete index theorems (discrete, because the index is integer-valued).

In 1977 Brüning (unpublished) defined Toeplitz operators on $S^{2n-1}$ whose symbols take values in a $W^*$-algebra factor of type $II_1$. Using the real-valued index map for Fredholm operators relative to a von Neumann algebra introduced by Breuer [6] he proved a continuous index theorem for such operators on $S^1$. The cases $n > 1$ and Toeplitz operators on $S^{2k-1} \times S^{2\ell-1}$ will be treated in the present paper.

2. THE PERIODICITY THEOREM

The main tool in the proof of our index theorems is the periodicity theorem for $W^*$-algebras. This will be stated shortly after some preliminary notations.

Throughout, $\mathcal{M}$ means a $W^*$-algebra factor of type $II_1$ (or $\mathcal{M} = \mathbb{C}$ to include the discrete case) with faithful normed trace
For $m \in \mathbb{N}$ let $\mathcal{M}_m$ denote the $W^*$-algebra of $m \times m$ matrices with entries from $\mathcal{M}$ and unit $1_m$. $\mathcal{M}_\infty$ their inductive limit, and $GL_m(\mathcal{M})$ and $GL_\infty(\mathcal{M})$ resp. $U_m(\mathcal{M})$ and $U_\infty(\mathcal{M})$ the appropriate regular resp. unitary group. On $\mathcal{M}_m$ we consider the trace

$$\text{tr}_m((x_{ij})_{i,j=1}^m) := \sum_{i=1}^m \text{tr}(x_{ii}).$$

Furthermore, we denote by $[\cdot]_\infty$ the classes of homotopic continuous mappings from a topological space into $GL_\infty(\mathcal{M})$. Otherwise, we use the notation of [12].

$GL_m(\mathcal{M})$ carries the structure of a Banach Lie group. Let $\omega$ denote the canonical left differential form and

$$\theta_k := \text{tr}_m(\omega^k) := \text{tr}_m(\omega \wedge \ldots \wedge \omega),$$

the exterior product $\phi_1 \wedge \phi_2$ of two differential forms $\phi_1$, $\phi_2$ of degree $k_1$ resp. $k_2$ being defined by

$$\phi_1 \wedge \phi_2(\xi_1, \ldots, \xi_{k_1+k_2}) := \sum_{\sigma \in S} \phi_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k_1)}) \phi_2(\xi_{\sigma(k_1+1)}, \ldots, \xi_{\sigma(k_1+k_2)}).$$

It is known ([4], III, 3.17, Example 2) that $f^* \omega = f^{-1} df$, and so

$$f^* \theta_{2n-1} = \text{tr}_m((f^{-1} df)^{2n-1})$$

for $f \in C^\infty(S^{2n-1}, GL_m(\mathcal{M}))$. Furthermore,

$$\int_{S^{2n-1}} f^* \theta_{2n-1}$$

is a $C^\infty$-homotopy invariant, as can be seen from one of the usual proofs of the Poincaré lemma. We then have

**Theorem 1.** For $n > 0$ and $f \in C^\infty(S^{2n-1}, GL_m(\mathcal{M}))$ define

$$(1) \quad \tau_n(f) := c_n \int_{S^{2n-1}} \text{tr}_m((f^{-1} df)^{2n-1}), \quad c_n := \frac{(-1)^{n-1}(n-1)!}{(2n-1)!(2\pi i)^n}.$$  

Then $\tau_n$ induces an isomorphism $\tau_n : \pi_{2n-1}(GL_\infty(\mathcal{M})) \to \mathbb{R}$, whereas the groups $\pi_{2n}(GL_\infty(\mathcal{M}))$ vanish for $n \geq 0$. 