A QUANTITATIVE MAXIMUM ENTROPY THEOREM FOR THE REAL LINE

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A bound $B$ is constructed for a large and natural class of entropy integrals on the real line. The construction involves Burg's maximum entropy theorem for the circle group and techniques from harmonic analysis. As the solution of an extension problem, the construction of $B$ leads to a further construction problem which can be viewed as a means of refining Krein's theorem on positive definite extensions.

INTRODUCTION

We shall prove a constructive maximum entropy theorem for the real line. The exact statement is given in Theorem 2.6. The essential feature of the theorem is an inequality of the form,

$$\int_{-\infty}^{\infty} \frac{\log G(\gamma)}{\pi(1+\gamma^2)} \, d\gamma \leq \int_{-\infty}^{\infty} \frac{\log S(\gamma)}{\pi(1+\gamma^2)} \, d\gamma,$$

where $S$ is a computable non-negative integrable function on the real line which extends given continuous data on an interval and where $G$ is any one of a large class of functions extending the same data.

The classical maximum entropy theorem for discrete data is due to Burg, and Theorem 2.1 is his result. In our view, and from a mathematical perspective, Theorem 2.6 is the continuous analogue of this theorem. There are also continuous analogues of Burg's theorem due to Dym and Gohberg [8] and Chover [5], as well as natural analogues dealing with linear

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fractional transformations \([2;3]\). Dym and Gohberg's theorem differs from Theorem 2.6 in using different logarithmic integrals than \((0.1)\), thereby necessitating another presentation including both hypotheses and proof, cf. [1] and [9] for an analysis and geometrical interpretation of [8]. Chover's theorem states a logarithmic inequality similar to \((0.1)\), but is non-constructive and probabilistic, and also has a different interpretation than Theorem 2.6.

Our notation is given in Section 1. Section 2 contains necessary definitions and discussion as well as statements of Burg's theorem (Theorem 2.1) and our result (Theorem 2.6). Theorem 2.6 is proved in several steps in Section 5; and we collect the little Fourier analysis required for the proof in Section 4. In Section 3 we focus on our chief hypothesis in Theorem 2.6 and pose a natural extension problem associated with a classical theorem due to Krein.

1. NOTATION

\(\mathbb{R}\) is the real line thought of as the time axis, and \(\hat{\mathbb{R}}\) is the real line, the dual group of \(\mathbb{R}\), thought of as the frequency axis. \(\mathbb{Z}\) designates the integers and \(\mathbb{T}_\omega = \hat{\mathbb{R}}/2\pi\mathbb{Z}\) is the compact group identified with the interval \([-\Omega,\Omega)\) for \(\Omega > 0\) fixed; we write \(\mathbb{T}\) instead of \(\mathbb{T}_\omega\). \(L^1(\mathbb{R})\) is the space of complex \((\mathbb{C})\)-valued Lebesgue integrable functions \(G\) on \(\hat{\mathbb{R}}\), normed by \(\|G\|_{L^1(\mathbb{R})} = \int |G(\gamma)| d\gamma\), where \(\int\) denotes integration over \(\hat{\mathbb{R}}\). \(L^1(\mathbb{T}_\omega)\) is the space of \(\mathbb{C}\)-valued \(2\Omega\)-periodic locally Lebesgue integrable functions \(F\) on \(\hat{\mathbb{R}}\), normed by \(\|F\|_{L^1(\mathbb{T}_\omega)} = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)| d\gamma\).

The Fourier transform of \(G \in L^1(\mathbb{R})\) is
\[
\mathcal{F}(G(t)) = \int G(\gamma) e^{-2\pi i t \gamma} d\gamma, \quad t \in \mathbb{R}.
\]

The Fourier series of \(F \in L^1(\mathbb{T}_\omega)\) is \(\sum \mathcal{F}(j) e^{2\pi ij \gamma/\Omega}\), summed