ON $\ell_p$-SUMMABILITY OF THE CHARACTERISTIC 
VALUES OF INTEGRAL OPERATORS ON $L_2(\mathbb{R}^N)$

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Let $B_p(L_2)$ ($0 < p < \infty$) denote the space of all compact linear operators $K$ on $L_2(\mathbb{R}^N)$ with the following property: the sum of $p$-th powers of all positive eigenvalues (repeated according to their multiplicity) of the nonnegative compact operator $(K^*K)^{1/2}$ converges. We prove a $B_p$-criterion for an integral operator $K$ on $L_2(\mathbb{R}^N)$, for $0 < p \leq 2$. We require that its kernel have sufficient fractional smoothness and decay at infinity. Our proof is based on a suitable factorization of the operator $K$ which involves fractional powers $(H + \lambda I)^{-\omega}$, $\lambda \geq 0$, $0 < \omega \leq 1$, of the resolvent of the selfadjoint elliptic operator $H = (-\Delta)^m + M_q^{2m}$ on $L_2(\mathbb{R}^N)$.

0. INTRODUCTION

In our recent paper (Takáč [22, Theorem 1.3]) we derived a simple formula for the resolvent $(H + \lambda I)^{-1}$, $\lambda > 0$, of the selfadjoint elliptic operator $H = (-\Delta)^m + M_q^{2m}$ on $L_2(\mathbb{R}^N)$, where $N \geq 1$ and $m \geq 1$ are integers, $\Delta$ is the Laplacian, and $M_q$ denotes the operator of pointwise multiplication by a positive continuous function $q$ on $\mathbb{R}^N$. The goal of the present paper is to generalize the main results of the previous one as well as those of Takáč [23]. In the latter (Theorem 1.6) we formulated a trace class criterion for an integral operator $K$ on $L_2(\mathbb{R}^N)$. In this criterion we stated sufficient conditions on the kernel $k(x, y), x, y \in \mathbb{R}^N$, of the integral operator $K$ which imply that $K$ is of trace class. These conditions require that the kernel $k(x, y)$ have both sufficient fractional smoothness and decay at infinity with respect to the $x$-variable. As a generalization of this criterion we will formulate a $B_p$-criterion ($0 < p \leq 2$) for an integral operator $K$ on $L_2(\mathbb{R}^N)$ which we state as Theorem 1.6 again. We recall that $B_1$ is the trace class, and $B_2$ is the Hilbert-Schmidt class (cf. Kato [11]). In this criterion we formulate sufficient conditions on the kernel $k(x, y), x, y \in \mathbb{R}^N$, of the integral operator $K$ which imply that $K$ is in $B_p(L_2)$. These conditions require that the kernel $k(x, y)$ have both sufficient fractional smoothness and decay at infinity with respect to both $x$- and $y$-variables. As a direct consequence of our Theorem 1.6, we state Corollary 1.7 which
shows overlapping between our results and those of Kamp, Lorentz and Rejto [8].
Finally we illustrate the optimality of our trace class criterion with Example 1.8.

As for the organization and methods of this paper, we state our main results as Theorems 1.3 through 1.6 in Section 1. We refer to Takáč [22] for the proofs of Prop. 1.2 and Theorem 1.3 (the formula for \((H + \lambda I)^{-1}\)) and to Takáč [23] for the proof of Theorem 1.5.

In Section 2 we prove Theorem 1.4. To find a necessary and sufficient condition for \(\alpha = 2m\omega\) and \(q\) that the operator \(A^{-1}_{\lambda,\omega} = (H + \lambda I)^{-\omega}\) on \(L_2(\mathbb{R}^N)\) be in \(\mathcal{B}_r(\mathcal{L}_2)\), for some \(r \in (0, \frac{2}{\omega}]\), we estimate the Hilbert-Schmidt norm of the operator integral

\[
(H + \lambda I)^{-\sigma} = \frac{\sin \sigma \pi}{\pi} \int_0^\infty (H + \lambda I + \mu I)^{-1} \mu^{-\sigma} d\mu
\]
on \(L_2(\mathbb{R}^N)\), where \(\sigma = \omega r/2 < 1\). This estimate is based on our earlier result (cf. Takáč [22, Theorem 1.4]) for \(\sigma = 1\).

In Section 3 we prove Theorem 1.6. We first factorize the integral operator \(K\) as \(K = A^{-1}_{\lambda,\omega}((A'_{\lambda,\omega'})^{-1} B)^*\), where \(B = A'_{\lambda,\omega'}(A_{\lambda,\omega} K)^*\), \(A_{\lambda,\omega} = ((-\Delta)^m + M^m_2 + \lambda I)^\omega\) and \(A'_{\lambda,\omega'} = ((-\Delta)^{m'} + M^m_{2}' + \lambda I)^{\omega'}\) with \(m', q'\) and \(\omega'\) satisfying the same hypotheses as \(m, q\) and \(\omega\). Namely, the smoothness and decay conditions imposed on the kernel \(k(x, y)\) of \(K\) guarantee that also the product \(B\) is a Hilbert-Schmidt operator on \(L_2(\mathbb{R}^N)\). Furthermore, we have \(A^{-1}_{\lambda,\omega} \in \mathcal{B}_r(\mathcal{L}_2)\) and \((A'_{\lambda,\omega'})^{-1} \in \mathcal{B}_{r'}(\mathcal{L}_2)\), for some \(r \in (0, \frac{2}{\omega}]\) and \(r' \in (0, \frac{2}{\omega'}]\). Finally we exploit the well-known fact that the product of three operators from \(\mathcal{B}_r(\mathcal{L}_2), \mathcal{B}_2(\mathcal{L}_2)\) and \(\mathcal{B}_{r'}(\mathcal{L}_2)\), namely \(K = A^{-1}_{\lambda,\omega} B^*((A'_{\lambda,\omega'})^{-1})^*\), is in \(\mathcal{B}_p(\mathcal{L}_2)\) where \(p = (\frac{1}{r} + \frac{1}{2} + \frac{1}{r'})^{-1}\) (cf. Gohberg and Krein [6, Chapter III, §7, Ineq. (7.5), p. 92], or Simon [17, Theorem 2.8, p. 31]). As usual, \(*\) denotes the adjoint.

In Section 4 we suggest and discuss several possible generalizations of our results. We also compare our results to those already known. As for our Theorems 1.3 and 1.4, we refer to Titchmarsh [24, §17.13, p. 182] for the idea of fractional integration (for \(N = 2, m = 1\)), and to Triebel [27, Section 6.6.1, Theorem 1, p. 423] for similar results. These methods (including ours) fail if \(A_{\lambda,\omega}\) is not selfadjoint on