INVERSION OF MATRICES WITH DISPLACEMENT STRUCTURE

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Inversion formulas and fast inversion algorithms for matrices the entries of which fulfill a difference equation are established. In that way the Gohberg/Semencul and Gohberg/Krupnik theorems and related results will be generalized.

1. INTRODUCTION

Let $D = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ be a nonsingular 2x2 matrix. An $m \times n$ matrix $A = [a_{ij}]_{0}^{m-1}n-1 \in C^{m \times n}$ is said to possess an $D$-structure if and only if

$$aa_{ik} + ba_{i-1,k} + ca_{i,k-1} + da_{i-1,k-1} = 0$$

for all $i = 1, \ldots, m-1$ and $k = 1, \ldots, n-1$.

The class of $m \times n$ matrices with a $D$-structure will be denoted by $Str_{mn}(D)$ or $Str(D)$. Special cases are the class of Toeplitz matrices, $D = D_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and Hankel matrices $D = D_H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. More general classes occur in polynomial regression problems on circles and straight lines (see below).

In the present paper we continue our investigation of [HR2], but the considerations here are essentially self-contained. The main aims are the construction of inversion formulas of the Gohberg/Semencul type and of fast inversion algorithms of the Levinson/Trench type [L], [T].

The classical Gohberg/Semencul formula establishes the inverse of a Toeplitz matrix from its first and last columns [GS], provided that the entry in the upper-left corner is non-zero. In Section 3 we shall establish the inverse of an $D$-structured matrix with the help of the solutions of two equa-
Ax_t = l_n(t) , Ax_s = l_n(s) (1.1)

where \( l_n(t) \) (or simply \( l(t) \)) is the vector \([1 \ t \ \ldots \ t^{n-1}]^T\) if \( t \in \mathbb{C} \), and \( l_n(\infty) = l(\infty) = [0 \ \ldots \ 0 \ 1]^T \). The choice for the classical formula is \( t = 0 \) and \( s = \infty \).

In Section 4 we construct the inverse matrix with the help of the solutions of the equations

\[ Ax = l(t) , \quad Ax' = l'(t) , \quad (1.2) \]

where \( l'(t) \) denotes the derivative of \( l(t) \) by \( t \) if \( t \neq \infty \) and \( l'(\infty) = [0 \ \ldots \ 0 \ 1 \ 0]^T \). The special case \( t = 0 \) and a Toeplitz will provide just the Gohberg/Krupnik formula [8K].

In [HRI] it is shown how to establish the inverse of a Toeplitz matrix with the help of certain fundamental equations. This construction works without any additional assumption and implies the other inversion formulas. In Section 5 we generalize this result to the case of matrices from \( \text{Str}(D) \).

The most natural approach seems to be, in our opinion, the construction of the inverse with the help of the solutions of a certain homogeneous equation. For Hankel and Toeplitz matrices this is implicitly contained in [HRI] and explicitly formulated in [A]. The generalization of this approach will be presented in Section 2.

Sections 6 and 8 are dedicated to fast inversion algorithms. In Section 6 we present an algorithm working for strongly nonsingular matrices based on the approaches of Sections 3 and 4. Let us note that there is an essential difference between the cases \( a \neq 0 \) and \( a = 0 \). In the case \( a \neq 0 \) we obtain a generalization of the Levinson/Trench algorithm for Toeplitz matrix inversion, whereas in the case \( a = 0 \) we obtain a generalization of the well-known Hankel matrix inversion algorithms. In Section 8 we present an inversion algorithm working for all nonsingular matrices from \( \text{Str}(D) \). As in the case of Hankel matrices (see [HRI]) this algorithm is based on some kernel structure properties. These properties will be formulated in Section 7.

Finally let us show, as an example, that D-structured