**Brief Note**

**Characteristics of sensitivity derivatives of static responses**

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**Abstract** Mathematical expressions for linear structures consisting of prismatic elements have been developed, which corroborate well-known facts: (a) nodal displacements depend on many design variables and (b) element stresses depend on fewer numbers of design variables. Even though the above expressions are valid for any type of linear finite element discretization, presently numerical experimentations are performed only for a frame structure.

1 Introduction

Design sensitivity analysis plays a central role in structural optimization. Examinations of optimization procedures indicate that calculation of sensitivity derivatives is the predominant contributor to time and memory requirements for structural optimization. Sensitivity analysis of structures has been dealt with extensively by Arora and Haug (1979), Arielman and Haftka (1986) and Haug et al. (1986). The work presented here is the development of mathematical expressions that are used to study the characteristics of sensitivity derivatives of static responses of linear structures consisting of prismatic elements. It is shown analytically that derivatives of nodal displacements with respect to several design variables are significant. Derivatives of stresses in an element with respect to very few design variables are significant. The latter is the key to the success of the fully stressed design (FSD) technique.

2 Sensitivity derivatives

A structure consisting of prismatic elements in which \( x_{nk} \) are design variables, where \( n = 1,2,\ldots \), is the number of elements and \( k = 1,2,\ldots \), is the number of dimensions describing the cross-section of the element, is considered. All vectors and matrices are in the global coordinate system of the structure and all are expanded to the total number of degrees of freedom of the structure.

2.1 Element end forces

The Element end forces (\( f_i \)) of the \( i \)-th element are given by the expression

\[
f_i = K_i d + \bar{f}_i ,
\]

where \( \bar{f}_i \) is the element fixed end forces and \( K_i \) is the stiffness matrix of the \( i \)-th element and \( d \) is the vector of nodal displacements. The gradient of \( f_i \) with respect to \( x_{nk} \) can be written as

\[
\nabla_{nk} f_i = \nabla_{nk} K_i d + K_i \nabla_{nk} d + \nabla_{nk} \bar{f}_i ,
\]

where \( \nabla_{nk} \) is a gradient operator with respect to \( x_{nk} \). It is known that

\[
K d = p ,
\]

where \( K \) is the global stiffness matrix of the structure and \( p \) is the vector of nodal loads,

\[
\nabla_{nk} d = K^{-1} \nabla_{nk} p - K^{-1} \nabla_{nk} K d .
\]

From (2) and (4),

\[
\nabla_{nk} f_i = \nabla_{nk} K_i d + K_i K^{-1} \nabla_{nk} p - K_i K^{-1} \nabla_{nk} K d + \nabla_{nk} \bar{f}_i .
\]

Taking the transpose of both sides, postmultiplying with \( d \) and simplifying,

\[
\nabla_{nk} f_i^T d = \nabla_{nk} p^T d_i + d^T \nabla_{nk} K_i [d - \bar{d}_i] + \nabla_{nk} \bar{f}_i^T d ,
\]

where \( \bar{d}_i = K^{-1} \bar{f}_i = K^{-1} K_i d \) and \( d_i \) is the vector of nodal displacements due to element end forces of the \( i \)-th element applied alone on the structure.

2.1.1 Case (i): \( n = i \) represents self-derivatives - the derivatives of the end forces of the \( i \)-th element with respect to its own cross-sectional dimensions. Then \( \nabla_{nk} K_i = \nabla_{ik} K_i, \nabla_{nk} K = \nabla_{ik} K_i \) and \( \nabla_{nk} p = \nabla_{ik} p_i \). From (6),

\[
\nabla_{ik} f_i^T d = \nabla_{ik} p_i^T d_i + d_i^T \nabla_{ik} K_i [d_i - \bar{d}_i] + \nabla_{ik} \bar{f}_i^T d_i , \quad (7a)
\]

or

\[
\nabla_{ik} f_i^T d_i = \nabla_{ik} p_i^T d_i + d_i^T \nabla_{ik} K_i [d_i - \bar{d}_i] + \nabla_{ik} \bar{f}_i^T d_i , \quad (7b)
\]

where \( d_i \) and \( \bar{d}_i \) are displacements of the \( i \)-th element due to total load \( p \) and \( f_i \), respectively.

From (7b) it is seen that the derivative of \( f_i \) with respect to its own \( k \)-th cross-sectional dimension, \( x_{ik} \), depends on \( d_i \) and \( \bar{d}_i \). However, the same with respect to the \( k \)-th cross-sectional dimension of the \( j \)-th element, \( x_{jk} \), depends on \( d_j \) only, as can be seen from (8b). The values of \( \bar{d}_i \) for all the elements surrounding the \( i \)-th element are expected to be generally small compared to \( \bar{d}_i \) and these will be very small, almost zero for elements far from the \( i \)-th element.
This observation is expected to be true for any finite element discretization. From these observations, it is expected that derivatives of member end forces of a member with respect to its own cross-sectional dimensions will be the most significant. They will be comparatively less significant with respect to cross-sectional dimensions of elements surrounding the element. Its derivatives with respect to the cross-sectional dimensions of far elements will be insignificant. To study the expected behaviour, derivatives of static responses of a frame structure is considered. The maximum normalized fibre stress of the $i$-th member is given as

$$
\sigma_i = \frac{P_i}{A_i \sigma_t} + \frac{M_{iy} y}{I_{yy} \sigma_b} + \frac{M_{iz} z}{I_{zz} \sigma_b},
$$

(9)

where $P_i$, $M_{iy}$ and $M_{iz}$ are the axial force and bending moments; $y$ and $z$ are the extreme fibre distances; $\sigma_t = 0.60 \sigma_y$ and $\sigma_b = 0.65 \sigma_y$; $\sigma_y$ is the yield stress. In general, $A_i$ and $I_i$ are the cross-sectional area and the moment of inertia of the $i$-th member, respectively.

2.2 Nodal displacements
The $j$-th displacement ($\Delta_j$) is given by

$$
\Delta_j = a^T \Delta_j = p^T K^{-1} \Delta_j = p^T d_{jv},
$$

(10)

where $\Delta_j$ is a virtual load vector consisting of unity load in the direction of $\Delta_j$ and $d_{jv}$ is the virtual displacement due to $\Delta_j$. The gradient of $\Delta_j$ and $d_{jv}$ with respect to $x_{nk}$ can be written as

$$
\nabla_{nk} \Delta_j = \nabla_{nk} p^T d_{jv} + p^T \nabla_{nk} d_{jv},
$$

(11)

and

$$
\nabla_{nk} d_{jv} = -K^{-1} \nabla_{nk} K d_{jv},
$$

(12)

From (11) and (12),

$$
\nabla_{nk} \Delta_j = \nabla_{nk} p^T d_{jv} - d \nabla_{nk} K d_{jv}.
$$

(13)

From (13), it is seen that derivatives of $\Delta_j$ with respect to many design variables are likely to be significant. For a particular element, its participation in resisting $\Delta_j$ depends on its stiffness, virtual nodal displacements due to $\Delta_j$ and nodal displacements due to $p$.

3 Numerical experimentations
For establishing the observations made regarding sensitivity derivatives of stresses and displacements, (a) the derivatives of the normalized maximum fibre stress of beams and (b) the sensitivity of nodal displacements with respect to all design variables are presented graphically.

3.1 Design independent loading
The details of a doubly symmetric frame structure, consisting of 56 members are shown in Fig. 1. The modulus of elasticity is $2.1 \times 10^{11}$ N/m$^2$, the yield strength is $250 \times 10^6$ N/m$^2$ and Poisson’s ratio is 0.3. Loads of 500 kN are applied in the X, Y and -Z directions at all the top four nodes. The members are tubular having an outer diameter of 750 mm and a thickness of 50 mm.

Sensitivity derivatives of $\sigma_4$, $\sigma_6$ and $\Delta_{6z}$ are being examined. Derivatives of $\sigma_4$ with respect to all diameters and thicknesses are shown in Fig. 2. It is seen that cross-derivatives of $\sigma_4$ with respect to cross-sectional dimensions of the members connected to it directly as well as all the members in the z-direction connected to fixed nodes are significant. Sensitivity derivatives of $\sigma_4$ are shown in Fig. 3. It is seen that cross-sensitivity derivatives are much lower in comparison to self-derivatives. This is as expected. The behaviour is found to be different for the z-direction members whose one end is constrained. Member end forces at both the ends are applied for all the members whose both ends are free to deform. Since that member undergoes max-