Optimal source control and resolution in nondestructive testing

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Abstract The paper considers the problem of damage detection arising in nondestructive testing. Applying currents on the boundary of a body and measuring the corresponding responses a conclusion should be made about the presence of damage inside the body.

The detection problem is formulated using a variational approach as a generalized eigenvalue problem. The maximal eigenvalue defines the accuracy of the measurements, which is necessary to detect this distribution of damage. The damage can be detected if there exists such a current in the set of the currents prescribed by the conditions of the experiment that generates perturbation on the boundary greater than the noise level in measurements.

To consider the worst case of detection, the damaged material should be distributed throughout the body in order to minimize the maximal eigenvalue of the spectral operator. An analytical estimate of the perturbation of the maximal eigenvalue is given, depending on the amount of damaged material.

1 Introduction
The present paper considers the problem of the detection of damaged parts inside a body when damaged material is characterized by different properties in comparison with an undamaged sample. By applying currents on the boundary and measuring the corresponding responses also on the boundary (or on the part of the boundary), a conclusion should be made about the presence of inclusions of damaged material inside the body when the measurements are of limited precision.

Two problems are formulated.
(1) The problem of the optimization of boundary sources - finding an excitation from a given set of functions which maximizes the measured difference in the responses. This problem is one of optimal boundary control and is formulated using a variational approach as a generalized eigenvalue problem. The maximal eigenvalue gives an estimate of the accuracy of the measurements needed to detect this particular distribution of damaged material inside the body.

(2) The problem of assured detection - finding an inclusion that is the most hidden or the most difficult to detect. This gives an estimate of a necessary accuracy of the measurements in the worst possible case of distribution of the damage. This second problem is one of minimizing the maximal eigenvalue, when the control is the property of the material in the interior of the body. An analytical solution for this problem is constructed based on the results of Cherkaeva and Cherkaev (1995) for the case when the sample is made of homogeneous material and the damaged material differs by its conductivity from the material of the sample. An analytical estimate is given of the total amount of damaged material inside a homogeneous sample, which is possible to detect with measurements of a given accuracy.

2 Variational formulation of the problem of detection in nondestructive testing
The problem of detection is a one of identification of the presence of a defect inside a body in a situation when measurements are of limited precision. It is assumed that currents can be applied on the boundary of the real body and on the boundary of the undamaged (or standard) sample. Measured are voltages or some other functional linearly dependent on the applied sources. The currents can be chosen from some set of functions describing physical constraints of the experiment, for example, that the energy of the applied source is limited or the electrodes are placed only on part of the boundary.

If the difference between the measured and standard values of the functional is greater than the level of noise in the measurements, which is assumed to be known, then the real sample certainly contains some defects inside. If the responses differ by less than possible noise in data, we can say nothing about the possible presence of an inclusion.

The optimization problem is to choose the excitations from a given set which would maximize the measured difference in responses, and to determine what damage inside the sample is possible to detect having measurements with a prescribed level of noise \( \eta \).

Let the body contain some parts of a damaged material inside, so that the change of the conductivity is described by the function \( \sigma \). We assume that we can inject a current \( f \), chosen from some set of functions \( f \in \mathcal{F} \) such that
\[
\mathcal{F} = \{ f : (Hf, f) = 1 \}.
\] (1)
Here selfadjoint operator \( H \) describes the constraints of the experiment for the set of the currents, and \((\cdot,\cdot)\) is a scalar product in an appropriate space of functions. We measure a difference in the responses given by a function \( g \), which is the value of a (linear with \( f \)) functional \( A_\sigma \) depending on the defect \( \sigma \) for the applied current \( f \).

The optimization problem for the optimal current is to find a function \( f \) in the set \( \mathcal{F} \) which maximizes the norm of the measured function \( g \). It can be written as a variational problem of the maximization of a ratio of energies, which has the form of the Rayleigh quotient functional
\[
J = \max_{f \in \mathcal{F}} \frac{\langle A_\sigma f, f \rangle}{\langle Hf, f \rangle}.
\] (2)
When $A_\sigma$ is the difference in the voltage measurements due to the same injected current, $A_\sigma = V_\sigma$, then $\langle V_\sigma f, f \rangle$ is the difference of the energies spent for injection of the same current $f$ into the real and the standard samples. For $A = V_\sigma^* V_\sigma$, the numerator of the functional $J$ is a squared norm of the measured difference between the voltages in the real and the standard samples. With $H$ equal identity operator, the norm of the applied currents is assumed to be unit; for this particular case the spectral problem of this type was formulated by Isaacson (1986) and Gisser et al. (1990). Assuming $H$ to be an operator mapping injected currents into the measured voltages in the real body, we restrict the energy of the current injected in the sample to be unit.

Suppose that the maximal value of this functional is $\lambda_0$. Then varying with respect to $f$ we obtain the generalized eigenvalue problem with the eigenvalue $\lambda_0$,

$$\langle A_\sigma f, f \rangle - \lambda_0 \langle H f, f \rangle = 0 \quad \text{and} \quad A_\sigma f = \lambda_0 H f . \quad (3)$$

The eigenvalue $\lambda_0$ defines the accuracy of the measurements which is necessary for the detection of the defect $\sigma$ using this type of experiment. Noise level $\eta$ in measurements should be less than $\lambda_0$,

$$\lambda_0 > \eta . \quad (4)$$

In this case the first eigenfunction of (3) provides a difference in measurements on the boundary greater than the noise level, and the presence of the defects can be identified.

If the currents can be injected only on a part of the boundary $\partial \Omega_1$ and the voltages are to be measured on the part $\partial \Omega_2$, then to account for this the scalar products in (2) and (3) can be taken as $\langle A_\sigma f, f \rangle_{\partial \Omega_2}$ and $\langle H f, f \rangle_{\partial \Omega_2}$.

For each particular application the problem can be specified to account for constraints of the experiment. We consider below two particular cases of the detectability problem (2)-(3) and derive corresponding generalized eigenvalue problems: for a case when the electrodes can be placed only on a part of the boundary $\Gamma, \Gamma \subset \partial \Omega$, and for a case when in order to detect a damage we compare the energies spent for injecting the same current into the real body and into the standard sample.

3 Spectral problem with boundary eigenfunctions

Let a body $\Omega$ with the boundary $\partial \Omega$ be filled with an isotropic material of the conductivity $\gamma$. We assume that arbitrary currents $f$ can be applied on the surface of the body and the corresponding responses are measured. The correspondence between the applied currents and the measured voltage responses is given by a linear operator $L_\gamma$ depending on the conductivity $\gamma$

$$\nabla \cdot \gamma \nabla u = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = f, \quad x \in \partial \Omega , \quad (5)$$

where the current $f$ satisfies the zero integral restriction: $\int_{\partial \Omega} f \, dx = 0$. The solution of (5) is unique up to a constant component, which is determined if a zero value of the function is prescribed; so we assume that $\int_{\partial \Omega} u \, dx = 0$.

Boundary voltages are given by the Neumann to Dirichlet operator $R_\gamma$, $L_2(\partial \Omega) \rightarrow L_2(\partial \Omega)$, which puts the correspondence between the injected currents and the measured voltages,

$$R_\gamma f = u \text{ on } \partial \Omega, \quad \text{such that } f = \gamma \frac{\partial u}{\partial n} . \quad (6)$$

This mapping is inverse to the Dirichlet to Neumann boundary map $A_\gamma$ which transforms the potential on the boundary to the values of the normal derivative (Sylvester and Uhlmann 1989).

Let the body contain some inclusions of damaged material inside so that its conductivity is described by the function $\gamma^*$. Assuming that the perturbation of the properties $\sigma = \gamma^* - \gamma$ is small, consider operator $L_{\gamma^*}$, which is a perturbed operator $L_\gamma$ (5). The function $\gamma^*$ describes the conductivity of the real medium with inclusions of damaged material, while the known unperturbed function $\gamma$ is the conductivity of the model medium.

Let an operator $V_\sigma$ be the difference in the boundary voltage measurements generated by the current $f$, corresponding to the media with the conductivities $\gamma^*$ and $\gamma$,

$$V_\sigma f = R_{\gamma^*} f - R_\gamma f , \quad (7)$$

with $R_\gamma$ defined in (6). Its norm is given by the first eigenvalue $\mu_1$,

$$\mu_1 = \max_{\|f\|=1} \|V_\sigma f\| . \quad (8)$$

3.1 The problem with data only on a part of the boundary

As an example, we consider a case when the injecting and measuring electrodes can be placed only on a part of the boundary $\Gamma, \Gamma' \subset \partial \Omega$. Using the approach developed, the detectability problem can be formulated as a generalized eigenvalue problem with the eigenvalues in mixed boundary condition.

The optimal currents injected on the part of the boundary $\Gamma$ that maximizes the difference in voltage responses $u = R_\gamma f$ and $w = R_{\gamma^*} f$ due to the media of conductivity $\gamma$ and $\gamma^*$, solve the following optimization problem:

$$J = \int (R_{\gamma^*} f - R_\gamma f)^2 \, dx \rightarrow \max_{f \in F_0} . \quad (9)$$

where we assume that $F_0 = \{ \|fX_\Gamma\|_{L_2(\partial \Omega)} = 1 \}$. Here $X_\Gamma$ is the index function of the domain $\Gamma$,

$$X_\Gamma(x) = \begin{cases} 1, & \text{if } x \in \Gamma \\ 0, & \text{if } x \in \partial \Omega / \Gamma \end{cases} . \quad (10)$$

The problem (9) is homogeneous; let its maximal value be $\mu$. Varying it and taking into account the differential constraints, we obtain a system of Euler Lagrange equations for the optimal potentials $u$ and $w$ and the Lagrange multipliers $p$ and $q$. The system of equations for the optimal $u$ and $w$ contains the Lagrange multipliers $p$ and $q$ in the mixed boundary condition

$$\nabla \cdot \gamma \nabla u = 0, \quad x \in \Omega, \quad (p + q)X_\Gamma - 2\mu \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega , \quad (11)$$

The system of equations for the Lagrange multipliers $p$ and $q$ contains the functions $u$ and $w$ in the boundary condition

$$\nabla \cdot \gamma \nabla p = 0, \quad x \in \Omega, \quad 2(w-u)X_\Gamma + \gamma \frac{\partial p}{\partial n} = 0, \quad x \in \partial \Omega , \quad (12)$$

$$\nabla \cdot (\gamma + \sigma) \nabla q = 0, \quad x \in \Omega, \quad 2(w-u)X_\Gamma - \gamma \frac{\partial q}{\partial n} = 0, \quad x \in \partial \Omega . \quad (13)$$