ON THE NONEXISTENCE OF A CONTINUOUS LINEAR OPERATOR
OF A CERTAIN FORM FROM $S$ TO $H(D)$

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In this note we show that a continuous linear operator of a certain form from the space $S$ of infinite sequences into the space $H(D)$ of analytic functions in a domain $D$ does not exist.

Let $D$ be an arbitrary domain on the complex plane, and let $\{a_k\}_{k=1}^{\infty} \in D$ be a fixed sequence of distinct points which has no limit points in $D$. Using the Weierstrass procedure (see [2]), one can construct for any sequence of numbers $\xi = \{\xi_k\}_{k=1}^{\infty}$ a function $f(z) = \xi_k$ analytic in $D$ such that $f(a_k) = \xi_k$ for $k = 1, 2, ...$. We denote by $S$ a linear topological space of sequences with the topology of coordinate convergence, and by $H(D)$ a linear topological space of functions analytic in $D$ with the topology of uniform convergence on every compact subset $K$ of $D$.

**Statement.** There does not exist a continuous linear operator $A$ from $S$ into $H(D)$ such that $A(\{\xi_k\}_{k=1}^{\infty}) = f_\xi(z)$ with $f_\xi(a_k) = \xi_k$, $k \geq 1$.

**Proof.** We assume on the contrary that such an operator exists. For each $k \geq 1$ we denote by $e_k \in S$ a sequence whose $k$-th component is unity and the remaining components are zero. Let $A(e_k) = \Phi_k(z)$. Since by definition $\Phi_k(a_k) = 1$, we have $\Phi_k(z) \neq 0$. Since $A$ is linear, we can write

$$A(\{\xi_k\}_{k=1}^{\infty}) = A \left( \sum_{k=1}^{n} \xi_k e_k \right) = \sum_{k=1}^{n} \xi_k \Phi_k(z).$$

The sequence of elements $s_n = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$ converges in $S$ to the element $\xi = \{\xi_k\}_{k=1}^{\infty}$. Therefore, the sequence of functions $\sum_{k=1}^{n} \xi_k \Phi_k(z)$ converges to the function $f_\xi(z)$ in $H(D)$. In other words

$$f_\xi(z) = \sum_{k=1}^{\infty} \xi_k \Phi_k(z),$$

where the series converges uniformly on every compact subset $K$ of $D$. In particular, for every sequence $\{\xi_k\}_{k=1}^{\infty} \in S$ and for any point $z \in D$ the series

$$\sum_{k=1}^{\infty} \xi_k \Phi_k(z)$$

converges...
converges to a finite value. Now, assuming that infinitely many functions \( \Phi_{kj} \) are not zero at some point \( z_0 \in D \), we can construct a sequence \( \{\xi_k\}_{k=1}^{\infty} \) as follows: \( \xi_{kj} = 1/\Phi_{kj}(z_0) \) for those \( k_j \) for which \( \Phi_{kj}(z_0) \neq 0 \), and \( \xi_k = 0 \) for the rest of the indices. Then for this sequence we would have

\[
\sum_{k=1}^{\infty} \xi_k \Phi_k(z_0) = 1 = \infty.
\]

(4)

Consequently, only a finite number of functions \( \Phi_k(z) \) can be different from zero at every point of the domain \( D \).

Let \( B \) be a closed disk belonging to \( D \). To every point \( z \) in \( B \) corresponds a finite subset \( \{ \Phi_{k_1}, \Phi_{k_2}, \ldots, \Phi_{k_j} \} \) of the set of functions \( \{ \Phi_k \}_{k=1}^{\infty} \) that includes only those functions that are not zero at \( z \). This correspondence maps the continuum set of points of the disk \( B \) into the set of all finite subsets of the countable set of functions \( \{ \Phi_k \}_{k=1}^{\infty} \), which is itself countable. Therefore, by the Luzin theorem there exists a subset \( \{ \Phi_{k_1}, \Phi_{k_2}, \ldots, \Phi_{k_j} \} \) whose preimage \( E \subset B \) in this mapping has cardinality of continuum. This means that all the functions belonging to the set

\[
\{ \Phi_{k_1}, \Psi_{k_2}, \ldots, \Phi_{k_j} \}
\]

are zero on the set \( E \). Since the set \( E \) contains infinitely many points, it has limit points in \( B \). From the uniqueness theorem it follows that all the functions from the set (5) are identically zero on \( B \). Since \( D \) is connected, all these functions are zero on \( D \). However, \( \Phi_k(a_k) = 1 \) holds for all \( k \geq 1 \). The resulting contradiction proves the statement.

**Corollary.** Let \( A \) be a continuous linear operator from \( S \) to \( H(D) \). Then the dimension of the image \( \text{Im}(A) \) of \( A \) is finite.

**Proof.** From the proof of the Statement it follows that if \( A(e_k) = \Phi_k(z) \), \( k \geq 1 \), then \( \Phi_k(z) \neq 0 \) only for a finite set of indices \( k_j \).

Our result can be related to interpolation in Banach spaces of analytic functions. Let \( D \) be a unit disk. For \( 1 \leq p \leq \infty \), we denote by \( H^p \) a Banach space of analytic functions \( f \) in \( D \) for which the functions \( f(0) = f(re^{i\theta}) \) are bounded in \( L^p \)-norm as \( r \rightarrow 1 \), Hoffman [3]. Let \( \{a_k\}_{k=1}^{\infty} \) be a sequence of points in the open unit disk. Following Shapiro and Shields [4], with each \( f \in H^p \) the sequence of weighted values

\[
T^p f = \{ f(a_k) (1 - |a_k|) \}^{1/p}_{k=1} \leq \infty
\]

(6)

is associated. The sequence \( \{a_k\}_{k=1}^{\infty} \) is called interpolating by functions of class \( H^p \) (or simply interpolating when \( p = \infty \)) if \( l^p \subset T^p H^p \), [4]. Here, as always, \( l^p \) denotes the space of sequences \( \{c_k\}_{k=1}^{\infty} \) for which \( \sum |c_k|^p < \infty \) for \( 1 \leq p < \infty \); and sup \( |c_k| < \infty \) for \( p = \infty \). It was shown by Carleson [1] that a necessary and sufficient condition for \( \{a_k\}_{k=1}^{\infty} \) to be an interpolating sequence is that there exists a positive number \( \delta \) such that

\[
\prod_{n \neq k} |(a_k - a_n)/(1 - \overline{a}_n a_k)| \geq \delta, \quad k \geq 1.
\]

(7)

Generalization of this result to spaces \( H^p \) was given in [4], where it was shown that for any given value of \( p \) \( (1 \leq p \leq \infty) \) \( T^p H^p = l^p \) if and only if the sequence \( \{a_k\}_{k=1}^{\infty} \) satisfies condition (7).