Exact analytical solutions for non-selfadjoint variable-topology shape optimization problems: perforated cantilever plates in plane stress subject to displacement constraints. Part I

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Abstract Lurie (1994, 1995a, b) proved recently that variable-topology shape optimization of perforated plates in flexure for non-selfadjoint problems leads to rank-2 microstructures which are in general nonorthogonal. An extension of the same optimal microstructures to perforated plates in plane stress will be presented in Part II of this study. Using the above microstructure, the optimal solution is derived in this part for cantilever plates in plane stress, which are subject to two displacement constraints. For low volume fractions the above solutions are shown to converge to the known truss solutions of Birker et al. (1994). The problem of homogenizing the stiffness of nonorthogonal rank-2 microstructures is also discussed.

1 Introduction

Optimal rank-2 microstructures for selfadjoint composite plate problems were derived by Lurie's research group in the early eighties. Lurie et al. (1982), for example, considered plates made out of two isotropic materials of different moduli of elasticity, having either one or the other material along any line segment normal to the middle plane of the plate. Gibiansky and Cherkaev (1984) considered plates involving two materials of differing specific weights and elastic moduli. These authors have shown that for a minimum compliance problem, the microstructure of a composite plate consists of two sets of mutually orthogonal ribs which have, respectively, first- and second-order infinitesimal widths (Fig. 1a). The same conclusion was later confirmed by Kohn and Strang (1986) and others.

Perforated plates can be regarded as a special, limiting case of composite plates, in which one material has both zero specific weight and zero elastic modulus. Following the above discoveries by the Lurie group, Rozvany, Olhoff, Bendsoe et al. (1985/87)

- derived the correct equivalent elastic constants (in the homogenization literature termed "rigidity tensor" $E_{ijkl}$) for rank-2 microstructures; and
- obtained, by means of optimality criteria methods, closed-form analytical solutions for axially symmetric plates with various load and support conditions.

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Whilst the above results were for perforated plates with a Poisson's ratio of zero value, Ong et al. (1988) extended the same results to nonzero Poisson's ratios, and Ong (1987) and Szeto (1989) to two-material composites. A review of this work was given by Rozvany (1989, pp. 349-351). Optimization based on orthogonal rank-2 microstructures involves at any point of the midplane of the plate three parameters: the two rib densities $a$ and $b$, and the orientation $\alpha$ of the first-order ribs with respect to a fixed direction ($z$ in Fig. 1a). These parameters are indicated in Figs. 1a and b, the latter showing the ribs in a “lumped” form.

Fig. 1. Optimal rank-2 microstructures for selfadjoint and non-selfadjoint plate problems

The optimal plate microstructure for non-selfadjoint problems was determined by Lurie (1994, 1995a, b) quite recently. It is also a rank-2 microstructure, but, in general, nonorthogonal (Fig. 1c). The number of unknown parameters at any point of the midplane is four (Figs. 1c and d):
the rib densities \((a, b)\) and the rib orientations \((\alpha, \beta)\) with respect to a fixed direction \((z)\).

The aim of this paper is to present exact analytical solutions of non-selfadjoint variable-topology shape optimization problems of structural mechanics. To the authors' knowledge, this is the first such attempt in the brief history of topology optimization. Its results could be used as test examples for numerical studies.

2 Homogenized elastic constants for truss-like continua and nonorthogonal rank-2 microstructures

In the homogenization procedure, we replace our nonhomogeneous microstructure cells, which consist of an isotropic material with a homogeneous but anisotropic cell of the same elastic properties on the macro-scale. Although homogenization involves rather advanced techniques in mathematical studies, here we shall use only elementary structural mechanics and Saint Venant's (1855) principle. This possibility was already demonstrated in the paper by Rozvany, Olhoff, Bendse et al. (1985/87), the derivation from which is reproduced here for illustrative purposes. Due to Saint Venant's principle, the local effects of the second-order ribs on the first-order ribs can be neglected, and hence it can be assumed that the first-order ribs are uniformly stressed in both directions within a cell. Since stress and specific cross-sectional areas referred to a unit width, we shall replace the cell width \(5\) with unity in the calculations that follow. Considering the case of zero Poisson's ratio \((\nu = 0)\) the homogenized strains \(\varepsilon_2\) in the direction of the first-order ribs \((y_2\) in Fig. 1b) is given by

\[
\varepsilon_2 = \frac{\sigma_2}{E}, \tag{1}
\]

where \(\sigma_2\) is the homogenized stress in the direction \(y_2\) and \(E\) is Young's modulus for the base material. In the direction of the second-order ribs \((y_1\) in Fig. 1b), we have essentially an equivalent bar of nonuniform cross-sectional area, the latter being \(\frac{b'}{b}\) over a length of \((1 - a)\) and unity over the length \(a\). It follows that we have the average strain

\[
\varepsilon_1 = (1 - a)\frac{\sigma_1}{bE} + a\frac{\sigma_1}{E} = \frac{a}{b} \frac{1 - a + ab}{b} \tag{2}
\]

in the direction of the second-order ribs.

Defining now the elastic constants (e.g. Bendse 1995) \(E_{1111}\) and \(E_{2222}\) as

\[
\sigma_1 = E_{1111}\varepsilon_1, \quad \sigma_2 = E_{2222}\varepsilon_2, \tag{3}
\]

relations (1) and (2) yield

\[
E_{1111} = \frac{bE}{1 - a + ab}, \quad E_{2222} = aE, \tag{4}
\]

which for \(\nu = 0\) agree exactly with those derived from the homogenization formulæ by Bendse (1995, p. 19, Eq. 1.16).

Since the flexural stiffness of even first-order ribs is a higher order infinitesimal, the shear stiffness \((E_{1212})\) of the microstructure in Fig. 1a is clearly zero

\[
E_{1212} = 0, \tag{5}
\]

which also agrees with Bendse's (1995, Eq. 1.16) results. Using similar elementary methods, the correct equivalent elastic constants were derived for \(\nu \neq 0\) and for two-material composites (Ong et al. 1988; Ong 1987; Szeto 1989). All these agree with results of later homogenization studies, which make little reference to them. Naturally, if the empty spaces in Fig. 1a are filled with a second material, then the shear stiffness in (5) becomes nonzero.

Considering now the nonorthogonal microstructure in Fig. 1c, the flexural stiffness of the second-order ribs is a very high order infinitesimal and hence a stress in the direction \(y_1\) would cause, theoretically, an infinitely large strain, giving

\[
E_{1111} = 0. \tag{6}
\]

This would appear to be highly undesirable, and nonoptimal, if we considered forces in the direction \(y_1\). The fact that the microstructure can still be optimal is explained by the following reasoning.

![Fig. 2. Rank-1 microstructures for trusses](image)

Optimal trusses for several load conditions are known to consist of nonorthogonal networks of members having an infinitesimal spacing. Such a network is essentially a rank-1 microstructure (Fig. 2a), which clearly also has a zero stiffness in direction \(y_1\). This is because the truss members (of first-order infinitesimal width) have a higher order infinitesimal flexural stiffness. For this reason, engineers like to assume hinges at truss joints. However, the truss in Fig. 2b definitely has a finite stiffness for the loads in the direction corresponding to \(y_1\), even if we assume hinges at the joints and an infinitesimal member spacing. This is because at each joint the forces can be resolved into two components in the member directions and transmitted to the supports. The members themselves have a finite axial stiffness of \(aE\) where \(\alpha\) is the cross-sectional area per unit truss width. It follows that in the case of nonorthogonal rank-1 and rank-2 microstructures, the homogenized stiffnesses should not be calculated in the traditional fashion in orthogonal directions, but in the member directions.

If we now consider the microstructure in Fig. 1c, the elastic constant \(E_{2222}\) in the \(y_2\) direction has still the value in (4), since the strain is invariably given by (1) in that direction. Moreover, \(E_{1111}\) in (4) also remains valid in the direction of the second-order ribs \((y_1^*\) in Fig. 1c), because both lengths \((1 - a)\) and \(a\) in (2) are multiplied by \(1/\sin(\beta - \alpha)\) and hence for a unit length in the \(y_1^*\) direction the strain in (2) does not change. This means that \(E_{1111}\) in (4) also applies to nonorthogonal microstructures and is independent of \(\alpha\) and \(\beta\) if we consider it in the direction of the second-order ribs. Having determined the relevant homogenized elastic constants, we can proceed with the solution of the problems to be considered. It is to be remarked that we have only employed elementary structural mechanics in obtaining the above results, since proofs of convergence are of relatively little practical importance to the engineer.