On the solution of structures involving elements with nonconvex energy potentials

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Abstract In this paper, a new method is proposed for the numerical treatment of problems involving elements with nonconvex energy potentials. Due to the nonconvexity, the problem may have more than one stable solution. The method is applied for the solution of structures with semirigid connections whose behaviour are described by means of a nonmonotone moment-rotation diagram. The method is based on the mathematical background of hemivariational inequalities which describe rigorously the introduced nonconvexities. Numerical examples demonstrate the properties and the applicability of the proposed numerical method.

1 Introduction
Elements involving nonconvex energy potentials appear in several mechanical problems. The nonconvexity of the energy potential appears as a result of the integration of a nonmonotone stress-strain or reaction-displacement law. Consider, for example, the nonmonotone reaction \( S_i \) - displacement \( u_i \) diagram of Fig. 1a which leads to the nonconvex energy potential of Fig. 1b. Similar is the situation with the sawtooth stress-strain law of Fig. 1c in reinforced concrete under tension (Sanlon's diagram) which leads to the potential energy function of Fig. 1d. The same effects may appear also in other problems of structural mechanics. A very common case is that of a structure with connections obeying a non-linear moment-rotation law. This type of behaviour appears in almost every kind of civil engineering structure (concrete, steel, composite or timber) and is a result of pure or incomplete "cooperation" between the various structural elements (which is sometimes intentional) and includes various local instability effects such as buckling, crashing and cracking. The importance of this behaviour for the structural integrity, led to the COST-C1 action "Semirigid Behaviour of Civil Engineering Structural Connections" of the EEC (Colson 1992; Wald 1994). Figure 1e gives the typical moment-rotation diagram of a steel beam-to-column connection. In most cases, the diagram has a significant softening branch leading to the nonconvex energy potential of Fig. 1f.

All these problems that involve nonmonotone, possibly multivalued laws between stresses and strains or reactions and displacements are expressed through hemivariational inequalities (Panagiotopoulos, 1983, 1985, 1988, 1991, 1993; Panagiotopoulos et al. 1984; Moreau et al. 1988), which are expressions of the principle of virtual work in inequality form. According to the theory of hemivariational inequalities, stable solutions of the problem (that correspond to stable equilibrium configurations) are the local minima of the potential or the complementary energy of the structure. However, there exist some mathematical solutions of the problem which do not lead to stable equilibrium configurations, such as local maxima and saddle points. All the solutions of the equilibrium problem, which are called "substationarity" points (Panagiotopoulos 1991; Rockafellar 1979), are solutions of the differential inclusion \( 0 \in \partial \Pi(v) \), where \( \Pi \) is the potential energy, \( v \) is the displacement vector and \( \partial \) denotes the generalized gradient of F.H. Clarke - R.T. Rockafellar (Rockafellar 1979; Clarke 1985).

The above variational methods have significant advantages over the classical methods of linear and nonlinear analysis that encounter forbidding difficulties both in the formulation and in the numerical approximation of problems involving nonmonotone, possibly multivalued stress-strain or reaction-displacement laws. However, from the numerical point of view, the determination of all local minima of a nonconvex function is still an open problem in the theory of numerical optimization (Fletcher 1990; Pardalos and Rosen 1987). For this reason, several methods suitable for special classes of mechanical problems (i.e. for special cases of nonconvex energy functions) have been developed lately (Tzaferopoulos and Panagiotopoulos 1994; Tzaferopoulos et al. 1995; Stavroulakis et al. 1996; Demyanov et al. 1996).

In this paper we present a new method which replaces a problem involving nonconvex energy functions by a sequence of problems involving convex ones. As we will see, this is equivalent with the approximation of the nonmonotone stress-strain law with a series of monotone ones. Thus, we succeed in extending the advantageous numerical treatment of convex problems, to problems involving nonconvex ones and for large numbers of unknowns. The same questions can possibly be treated by the direct use of a nonconvex programming (NCP) algorithm, but then no large scale problem can be solved. Moreover the NCP algorithms do not have the speed, the convergence rate and the robustness of the convex programming algorithms.

Here, we will present the method by applying it in order to solve structures involving nonmonotone moment-rotation relationships. This problem was selected because it can be expressed in a relatively simple form and because it is easily approximated using the well-known elastoplastic law instead of the nonmonotone moment-rotation law. The results obtained can be easily extended in order to include a large class of problems involving nonmonotone laws (Mistakidis and Panagiotopoulos 1993, 1994).

In the second section, the method is presented in a gen-
eral form while in the third section it takes a form suitable for the special engineering applications treated here. In the fourth section the principles of the algorithm are demonstrated by solving an appropriately chosen simple example. In this model problem, several properties of the iterative scheme will be presented such as the numerical treatment of multiple equilibrium configurations. Finally, in the fifth section, a real engineering application is presented.

2 Presentation of the algorithm

Let us consider a structure \( \Omega \) having members which obey a nonmonotone stress-strain law. The boundary \( \Gamma \) of \( \Omega \) consists of the nonoverlapping parts \( \Gamma^u \) and \( \Gamma^F \), where the displacements and the forces are prescribed, respectively. We assume that the behaviour of \( \Omega \) is governed by a nonconvex energy law of the form

\[
\sigma = \partial \omega(\varepsilon),
\]

where \( \sigma = \{\sigma_{ij}\} \) and \( \varepsilon = \{\varepsilon_{ij}\} \) are the stress and strain tensors, respectively, and \( \omega(.) \) is a nonconvex strain energy functional. Relation (1) is by definition equivalent to the following hemivariational inequality (Panagiotopoulos 1993):

\[
\omega^0(\varepsilon, \varepsilon^*) 
\geq \sigma_{ij} [\varepsilon_{ij}(v) - \varepsilon_{ij}(u)] \quad \forall \varepsilon(v) \in \mathbb{R}^6,
\]

where \( \omega^0(\varepsilon, \varepsilon^*) \) denotes the Clarke directional derivative (Clarke 1983) of functional \( \omega(.) \) defined by

\[
\omega^0(\varepsilon, \varepsilon^*) = \limsup_{h \to 0} \frac{\omega(\varepsilon + \lambda \varepsilon^* + h) - \omega(\varepsilon + h)}{\lambda},
\]

Under the assumption of small deformations, the application of the principle of virtual work for the whole structure yields the relation

\[
\int_\Omega \sigma_{ij} [\varepsilon_{ij}(v) - \varepsilon_{ij}(u)] d\Omega + \int_{\Gamma^F} F_i (v_i - u_i) d\Gamma \quad \forall \varepsilon \in \mathcal{U}_{ad},
\]

where \( f = \{f_i\} \) is the body force vector, \( F = \{F_i\} \) is the given force on \( \Gamma^F \) and \( \mathcal{U}_{ad} \) denotes the kinematically admissible subset of the displacements (i.e. \( \mathcal{U}_{ad} = \{v|v_i = u_i \text{ on } \Gamma^u\} \)). Combining (2) and (4) we are led to the following problem. Find \( u \in \mathcal{U}_{ad} \) satisfying the inequality

\[
\int_\Omega \omega^0 [\varepsilon(u), \varepsilon(v - u)] d\Omega \geq \int_\Omega f_i (v_i - u_i) d\Omega + \int_{\Gamma^F} F_i (v_i - u_i) d\Gamma, \quad \forall \varepsilon \in \mathcal{U}_{ad}.
\]

Relation (5) is a hemivariational inequality due to the appearance of the energy variation term \( \int_\Omega \omega^0(\varepsilon, \varepsilon^*) d\Omega \) and expresses the principle of virtual work for the case of problems involving nonconvex energy potentials (Panagiotopoulos 1985, 1993).

Under certain conditions (cf. e.g. Naniewicz and Panagiotopoulos 1995) (5) is equivalent to the following stationarity problem.

Find \( u \in \mathcal{U}_{ad} \) such that the potential energy of the structure

\[
\Pi_{nc}(v) = \int_\Omega \omega(v) d\Omega - \int_\Omega f_i v_i d\Omega - \int_{\Gamma^F} F_i v_i d\Gamma,
\]

is stationary at \( v = u \).

Let us now suppose that instead of the nonconvex strain energy functional \( \omega(.) \), the convex energy functional \( p(.) \) holds on \( \Omega \). Then, instead of (2) the following inclusion holds:

\[
\sigma \in \partial p(\varepsilon),
\]

Relation (7) is by definition equivalent to the following variational inequality:

\[
p(\varepsilon(v)) - p(\varepsilon(u)) \geq \sigma_{ij} [\varepsilon_{ij}(v) - \varepsilon_{ij}(u)] \quad \forall \varepsilon(v) \in \mathbb{R}^6.
\]

According to Panagiotopoulos (1985) an equilibrium configuration of \( \Omega \) can be obtained by the following variational inequality.

Find \( u \in \mathcal{U}_{ad} \) such as to satisfy the inequality

\[
\int_\Omega p(\varepsilon(v)) - p(\varepsilon(u)) d\Omega \geq \int_\Omega f_i (v_i - u_i) d\Omega + \int_{\Gamma^F} F_i (v_i - u_i) d\Gamma, \quad \forall \varepsilon \in \mathcal{U}_{ad}.
\]

Fig. 1. Nonmonotone laws and the corresponding nonconvex superpotentials