THE SPECTRUM OF A VOLterra COMPOSITION OPERATOR

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Lyubic's conjecture, that the Volterra composition operator of equation (1) is quasinilpotent iff condition (2) holds, is established and is generalized to Volterra composition operators on $L^p[0,1]$.

Let $\varphi : [0,1] \times [0,1]$ be continuous with $\varphi(0) = 0$ and define the bounded linear operator $V_\varphi : C[0,1] \to C[0,1]$ by:

$$V_\varphi f(x) = \int_0^x f(t) dt.$$  \hspace{1cm} (1)

Lyubic noted in [1] that $V_\varphi$ is quasinilpotent if

$$\varphi(x) < x \text{ for all } x \text{ in } [0,1] \hspace{1cm} (2)$$

and conjectured that the sufficient condition (2) is also necessary.

THEOREM 1. Let $\varphi : [0,1] \times [0,1]$ be continuous. The operator $V_\varphi$ defined on $C[0,1]$ by equation (1) is quasinilpotent if and only if condition (2) holds.

PROOF. Two basic properties of $V_\varphi$ are:

$$|V_\varphi f| \leq V_\varphi |f| \hspace{1cm} (3)$$

$$V_\varphi f \leq V_\varphi g \text{ if } f \leq g \hspace{1cm} (4)$$

Repeated application of (3) and (4) shows that

$$|V_\varphi^n f| \leq V_\varphi^n |f| \leq |f| V_\varphi^n \hspace{1cm} (5)$$
and therefore

\[ |V_p^n| = |V_p^{n+1}|. \]  

(An example of the use of equation (6) is in showing that, for \( \phi(x) = x^\alpha, \)  
\( 0 < \alpha < 1, \) \( V_p \) has spectral radius \( 1-\alpha \)). If \( \rho \) is also a continuous map  
of \([0,1]\) into itself with

\[ \rho \leq \phi \]  

then \( V_p^n \leq V_p V_p^{n-1} \leq \cdots \leq V_p \), and thus \( |V_p^n| \leq |V_p^n| \). From the standard  
formula for the spectral radius \( r \) it follows that

\[ r(V_p) \leq r(V_p). \]  

The well-known fact that the Volterra operator

\[ Vf(x) = \int_0^x f(t)dt \]  
is quasinilpotent, together with equation (8), shows that condition (2) is  
sufficient for \( V \) to be quasinilpotent.

The computational details which arise in showing that condition (2)  
is necessary are simplified by considering an analogous operator on a  
different Banach space. Let \( \phi : [0,1] \to [0,1] \) be a measurable function and  
define a bounded linear operator \( U_\phi : L^2[0,1] \to L^2[0,1] \) by

\[ U_\phi f(x) = \int_0^x \phi(t)f(t)dt \]  

Since properties (3) and (4) hold for \( U_\phi \) and \( |f| \leq \|f\| \) a.e.,

\[ |U_\phi^n| = |U_\phi^{n+1}|. \]  

For a continuous \( \phi \) equations (6) and (10) imply