Schatten Class Hankel Operators on the Bergman space

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In this paper we characterize Hankel operators $H_f$ and $H_\gamma$ on the Bergman spaces of bounded symmetric domains which are in the Schatten $p$-class for $2 \leq p < \infty$ and $f$ in $L^2$ using a Jordan algebra characterization of bounded symmetric domains and properties of the Bergman metric.

Let $\mathcal{D}$ be a bounded symmetric (Cartan) domain of rank $r$ with its standard (Harish-Chandra) realization in $\mathbb{C}^n$ ([8],[7],[12]). We may assume that $\mathcal{D}$ is circled, irreducible and contains the origin $0 \in \mathbb{C}^n$. Let $G$ denote the connected component of the biholomorphic automorphism group of $\mathcal{D}$ and $K$ its isotropic group of $0$ in $G$. Then $\mathcal{D} = G/K$. For a suitable subspace $C^r$ of $\mathbb{C}^n$, as in [8] $\mathcal{D} \cap C^r$ is a polydisk $\mathcal{D}^r$ and $\mathcal{D}$ can be written as a union of polydisks $K\mathcal{D}^r$, where $K$ is considered as a subgroup of $U(n)$ of $\mathbb{C}^n$. So we have

$$\mathcal{D}^r \hookrightarrow \mathcal{D} \hookrightarrow \mathbb{C}^n$$

where $i: \mathcal{D}^r \hookrightarrow \mathcal{D}$ is a holomorphic embedding.

For $d\Lambda(w)$ the usual Euclidean volume measure on $\mathbb{C}^n=\mathbb{R}^{2n}$, normalized so that $A(\mathcal{D})=1$, we consider the Hilbert space of square-integrable complex-valued functions $L^2 = L^2(\mathcal{D},d\Lambda)$ and the Bergman space $L_a^2 = L^2_a(\mathcal{D},d\Lambda)$ of holomorphic functions in $L^2$. Since the evaluation at any fixed point of $\mathcal{D}$ is a bounded functional on $L_a^2$, there is a function $K(z,w)$ in $L_a^2$ such that

$$f(z) = \langle f, K(z, \cdot) \rangle$$

for all $f$ in $L_a^2$.

In fact for any orthogonal basis $\{e_n(w)\}$, $K(z,w)$ can be represented as

$$K(z,w) = \sum_{k=1}^{\infty} e_n \overline{e_n}(w)$$

where the sum converges pointwise to $K(z,w)$.

We recall that the Bergman metric $H_z(u,v)$ for $z$ in $\mathcal{D}$ and $u$, $v$ in $\mathbb{C}^n$ is defined by

$$H_z(u,v) = \sum_{i,j} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j} \log K(z,z) u_i \overline{v_j}$$

Then $\mathcal{D}$ is a complete Hermitian symmetric space of noncompact type with the Bergman metric which gives the usual topology on $\mathcal{D}$. By definition, the Bergman distance $\beta(z,w)$ is given by

$$\beta(z,w) = \inf \int_0^1 \sqrt{H_{\tau(t)}(\gamma'(t),\gamma'(t))} \, dt$$

where the inf is taken over all geodesics in $\mathcal{D}$ which connect $z$ and $w$. 
For a bounded symmetric domain, for fixed \( w \) in \( \mathcal{D} \), as \( z \) goes to the topological boundary,

\[
K(z, z) \to +\infty
\]

and

\[
\beta(z, w) \to +\infty.
\]

Moreover \( K(z, w) \) and \( \beta(z, w) \) have the following invariance properties

\[
K(ka, kb) = K(a, b)
\]

for all \( k \) in \( K \) and

\[
\beta(ga, gb) = \beta(a, b)
\]

for all \( g \) in \( G \). For each \( a \) in \( \mathcal{D} \), there is a biholomorphic automorphism \( \phi_a \) of \( \mathcal{D} \) (\( \phi_a \) in \( G' \)) with the properties

1. \( \phi_a(a) = 0 \)
2. \( \phi_a \circ \phi_a = Id. \)

\( \phi_a \) is determined uniquely up to composition with an element of \( K \).

An operator \( T \) on Hilbert space \( H \) is said to be in the Schatten p-class if \( T^*T \) is compact

\[
\sum_{i=1}^{\infty} s_i^p < \infty
\]

where \( (T^*T)^{1/2} = \sum_{i} s_i e_i \otimes e_i \) if \( \{e_i\} \) are an orthogonal basis of \( H \). We use \( S_p \) to denote the set of all operators in Schatten p-class for \( p > 0 \).

Let \( P \) be the self-adjoint projection from \( L^2 \) onto \( L^2_0 \). For \( f \) and \( g \) in \( L^2 \), we consider the multiplication operator \( M_f \) on \( L^2 \) given by \( M_fg = fg \) and the Hankel operator \( H_f \) on \( L^2_0 \) given by \( H_f = (I - P)M_fP \) and the Toeplitz operator \( T_f \) on \( L^2_0 \) given by \( T_f = PM_fP \). The commutator \([M_f, P] = M_fP - PM_f\) is densely defined on \( L^2 \) and it is easy to check that

\[
[M_f, P] = H_f \bigoplus (-H_f^*).
\]

So studying the properties of \([M_f, P]\) is equivalent to studying the properties of both \( H_f \) and \( H_f^* \).

The aim of this paper is to characterize those functions \( f \) such that both \( H_f \) and \( H_f^* \) are in \( S_p \) for \( 2 \leq p < +\infty \). We prove a conjecture of K.Zhu, proved by him for the case of unit ball in \([14]\). It was shown in \([1]\) that for holomorphic function \( f \) on the unit disk \( D \), \( H_f \) is in \( S_p \) for \( 1 \leq p < \infty \) if and only if \( f \) is in the Besov space \( B_p \). We will present two proofs. One proof uses Jordan theoretic characterization of bounded symmetric domains. i.e. every bounded symmetric domain \( \mathcal{D} \) can be realized as the open unit ball of a uniquely determined Jordan triple system \( V \approx C^n \) for the so called spectral norm ([10],[12]). The second proof will reduce the problem to the polydisk case using the fact that a holomorphic mapping between two bounded symmetric domains decreases the Bergman metric in some sense. So inspired by Zhu’s method in \([14]\), we can completely prove his conjecture on the bounded symmetric domain.