ALGEBRAS GENERATED BY A WEIGHTED SHIFT

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Techniques of Olin and Thomson and of Chevreau, Pearcy and Shields are used to prove the following: If $T$ is an injective weighted shift on $H$ with $r(T) = \|T\|$ then for each $L \in (a(T), \text{weak-}* *)$, there exist $f, g \in H$ so that $L(A) = \langle Af, g \rangle$ for all $A \in a(T)$. Thus the map $i(A) = A$ is a homeomorphism from $(a(T), \text{weak-}*)$ onto $(a(T), \text{WOT})$.

INTRODUCTION

Let $H$ be a separable Hilbert space and $B(H)$ the algebra of bounded linear operators on $H$. Recall $B(H)$ is the dual of the trace class operators on $H$, $B_1(H)$, and is therefore endowed with a weak-star topology:

$$A_i \to 0 \text{ weak-}* \text{ provided}$$

$$\text{trace } (A_i B) \to 0 \text{ for each } B \in B_1(H).$$

For $T \in B(H)$, let $a(T)$ denote the weak-* closed algebra generated by $T$ and $I$. Then $a(T)$ is the dual of the Banach space

$$a(T)_* = B_1 / a(T)$$

where $a(T) = \{ B \in B_1 | \text{trace } (AB) = 0 \text{ for each } A \in a(T) \}$. Let $\| \| \|_*$ denote the quotient norm on $a(T)_*$, the predual of $a(T)$. $a(T)_*$ may also be characterized as

$$a(T)_* = (a(T), \text{weak-}*)^*.$$

See [1] for a discussion of the results above.

In [4], Olin and Thomson have refined the techniques of Brown to show that if $S$ is subnormal on $H$ then, for each element $L$ of $(a(S), \text{weak-}*)$ there exists vectors $a$ and $b$ in $H$ so that for each $A \in a(S)$

$$L(A) = \langle Aa, b \rangle.$$
Using the techniques of Olin and Thomson and of Chevreau, Pearcy and Shields [2], we obtain the analogous result for weighted shifts.

**THEOREM I:** Let $T$ be an injective unilateral weighted shift on $H$ so that $\|T\| = r(T)$. Then, for each $L \in \sigma(T)_*$, there exist $x$ and $y$ in $H$ so that

$$L(A) = \langle Ax, y \rangle$$

for each $A \in \sigma(T)$. Moreover, if $\varepsilon > 0$, $x$ and $y$ may be chosen so that

$$\|x\|, \|y\| \leq (1 + \varepsilon) \|L\|^{1/2}.$$

**THEOREM I':** If $T$ is an injective bilateral weighted shift on $H$, if $\|T\| = r(T)$, then the conclusion of I again holds.

**NOTATION AND PRELIMINARIES**

We denote the spectrum of an operator $T$ by $\sigma(T)$. (Similarly, $\sigma_a(T), \sigma_e(T)$ denote respectively the approximate point spectrum and essential spectrum of $T$, and so forth.) The spectral radius of $T$ is denoted by $r(T)$.

An excellent exposition of the theory of weighted shifts may be found in Shields [6], some of whose results are stated below.

Let $(\beta(n))_{n=-\infty}^{\infty}$ be a sequence of positive numbers so that $\beta(0)=1$, and define sequence spaces

$$L^2(\beta) = \{f(z) : \sum_{n=-\infty}^{\infty} |\hat{f}(n)z^n| \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \beta^2(n) < \infty\}$$

and

$$H^2(\beta) = \{f(z) : f(n) = 0 \text{ for each } n < 0 \}.$$  

Then $L^2(\beta)$ and $H^2(\beta)$ are Hilbert spaces with inner product

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n) \beta^2(n).$$

Notice the functions $f_n(z) = z^n$ form an orthogonal basis for $L^2(\beta)$ and $\|f_n\| = \beta(n)$. Also, for $f \in L^2(\beta)$, the $n$th Fourier coefficient of $f$ is

$$\hat{f}(n) = \langle f, f_n \rangle \beta^{-2}(n).$$

**THEOREM A [6]:**

(i) If $T$ is an injective unilateral shift then there exists a sequence $(\beta(n))_{n=0}^{\infty}$ so that $T$ is unitarily equivalent to $M_z$ on $H^2(\beta)$ where

$$M_zf = \sum_{n=0}^{\infty} \hat{f}(n)z^{n+1}.$$