THE NUMERICAL RANGE OF ELEMENTARY OPERATORS

A. SEDDIK

For n-tuples $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ of operators on a Hilbert space $H$, let $R_{A,B}$ denote the operator on $L(H)$ defined by $R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$. In this paper we prove that

$$\co \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in W(A), (\beta_1, ..., \beta_n) \in W(B) \right\} \subseteq W_0(R_{A,B})$$

where $W$ is the joint spatial numerical range and $W_0$ is the numerical range. We will show also that this inclusion becomes an equality when $R_{A,B}$ is taken to be a generalized derivation, and it is strict when $R_{A,B}$ is taken to be an elementary multiplication operator induced by non scalar self-adjoints operators.

Introduction.

All operators considered here are bounded operators on a complex Hilbert space $H$. The collection of operators in $H$ is denoted by $L(H)$. We denote by $\text{tr}$ the trace map on the Banach space $(C_1(H), |||\cdot|||_1)$ of operators of class trace on $H$; and if $M \subset C$, we denote by $M^-$ and $\co M$ respectively the closure and the convex hull of $M$. If $\mathcal{A}$ is a complex unital Banach algebra and $A \in \mathcal{A}$, we denote by $W_0(A)$, the numerical range of $A$ given by:

$$W_0(A) = \{ f(A) : f \in \mathcal{P}(A) \}$$

where $\mathcal{P}(A) = \{ f \in \mathcal{A} : f(I) = ||f|| = 1 \}$ is the set of all states on $\mathcal{A}$. It is known that $W_0(A)$ is convex and compact, this result follows at once from the corresponding properties of the set of states. $A$ is called Hermitian if $W_0(A)$ is real. If $\mathcal{A} = L(H)$, then $W_0(A)$ is the closure of the usual numerical range $W(A)$ of $A$, where $W(A) = \{ (Ax, x) : x \in H, ||x|| = 1 \}$, this result follows immediately from [2] and [6]. For more details, see [3] and [8].

For n-tuples $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ of operators on $H$, we define:

(i) the joint spatial numerical range of $A$ (see [5]) by:

$$W(A_1, ..., A_n) = \{ (A_1 x, x), ..., (A_n x, x) : x \in H, ||x|| = 1 \}$$
(ii) the elementary operator $R_{A,B} : L(H) \to L(H)$ by:

$$\forall X \in L(H) : R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$$

For $A, B \in L(H)$, we also define the particular elementary operator:

(iii) the left multiplication operator $L_A : L(H) \to L(H)$ by:

$$\forall X \in L(H) : L_A(X) = AX$$

(iv) the right multiplication operator $R_B : L(H) \to L(H)$ by:

$$\forall X \in L(H) : R_B(X) = XB$$

(v) the generalized derivation $\delta_{A,B} = L_A - R_B$ induced by $A, B$.

(vi) the inner derivation $\delta_A = L_A - R_A$ induced by $A$.

(vii) the elementary multiplication operator $M(A, B) = L_AR_B$ induced by $A, B$.

For $x, y \in L(H)$, we define the operator $(x \otimes y)$ on $H$ by:

$$\forall z \in H : (x \otimes y)(z) = \langle z, y \rangle x$$

Curto [4] proved that, if $A$ and $B$ are $n$-tuples of commuting operators on $H$ then:

$$\sigma(R_{A,B}) = \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, \ldots, \alpha_n) \in \sigma_T(A), (\beta_1, \ldots, \beta_n) \in \sigma_T(B) \right\}$$

where $\sigma_T$ is the joint spectrum (the spectrum in the sense of J. L. Taylor, see [9]).

In this note, we give a similar result with the numerical range and the joint spatial numerical range without the assumption of the commutativity, more precisely we will show that:

$$co \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, \ldots, \alpha_n) \in W(A), (\beta_1, \ldots, \beta_n) \in W(B) \right\} \subset W_0(R_{A,B})$$

and we obtain that this inclusion becomes an equality when $R_{A,B}$ is taken to be a derivation, and it is strict when $R_{A,B}$ is taken to be an elementary multiplication operator induced by non scalar self-adjoints operators.

**Theorem 1.** Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be $n$-tuples of operators on $H$. Then we have:

$$co \left\{ \sum_{i=1}^{n} \alpha_i \beta_i : (\alpha_1, \ldots, \alpha_n) \in W(A), (\beta_1, \ldots, \beta_n) \in W(B) \right\} \subset W_0(R_{A,B})$$

**Proof.** The proof of this Theorem is based on the construction of a special state on $L(L(H))$, we perform as follows: for $x, y \in H$ such that: $\|x\| = \|y\| = 1$, we define the linear functional $f$ on $L(L(H))$ by:

$$\forall F \in L(L(H)) : f(F) = tr [(y \otimes x) F(x \otimes y)]$$