By an oversight on the part of the authors this section was not included in the paper previously published in Integral Equations Operator Theory, volume 14/4 (1991), 466–500.

3. INTERPOLATION VIA REPRODUCING KERNELS

The theory of reproducing kernel spaces of holomorphic functions given in Sections 1 and 2 will now be applied to the interpolation problem (IP). This approach to solve interpolation problems was initiated by Dym in [Dy1,2]. As mentioned in the Introduction, the method developed in this paper has several points of contact with Dym’s work.

We build from the data of the interpolation problem (IP) the following finite dimensional, resolvent invariant space \( \mathfrak{M} \) of rational functions:

\[
\mathfrak{M} = \{ \frac{\mathbf{W}^q_{jq}}{(Z - \ell)^{-l}} : c, \ell \in \mathbb{C}^r \}
\]

where the \( n \times r \) matrices \( \mathbf{V}, \mathbf{W} \) and the \( r \times r \) matrix \( Z \) are all associated with the data of (IP) and defined in the Introduction. In order to apply the results of Section 2 to our interpolation problem, we connect the notations used there, cf. (2.3)–(2.6), with the ones used in the formulation of (IP) by setting the \( 2n \times 1 \) vectors \( \mathbf{c}_{jq} \) equal to

\[
\mathbf{c}_{jq} = \left( \begin{array}{c} \mathbf{W}^q_{jq} \\ \mathbf{V}^q_{jq} \end{array} \right)
\]

so that \( \mathbf{C} \) is the \( 2n \times r \) matrix

\[
\mathbf{C} = \left( \begin{array}{cccc} c_{10} & c_{11} & \cdots & c_{1r_1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m0} & \cdots & \cdots & c_{mr_m} \end{array} \right) = \left( \begin{array}{c} \mathbf{W} \\ \mathbf{V} \end{array} \right).
\]

Thus the linear space \( \mathfrak{M} \) can be written as

\[
\mathfrak{M} = 1.s. \{ F_{jq}(\ell) \mid j = 1,2, \ldots, m, \ q = 0,1, \ldots, r_j \}
\]

where \( F_{jq}(\ell) \), the \( q \)-th element of the \( j \)-th chain, is given by

\[
(3.1) \quad F_{jq}(\ell) = \sum_{h=0}^{q} (\ell - \bar{w}_j)^{-q+1+h} c_{jqh}.
\]

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From the formula
\[ T(\tau) = \left( F_{10}(\tau):F_{11}(\tau):\ldots:F_{1r_1}(\tau):\ldots:F_{m0}(\tau):\ldots:F_{mr_m}(\tau) \right) = -C(Z-\tau)^{-1} \]

it easily follows that \( \mathfrak{M} \) is resolvent invariant. Except when explicitly stated otherwise, we fix in this section the signature matrix \( J \) and set it equal to
\[ J = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}. \]

Note that now the Lyapunov equation (0.6) can be rewritten as
\[ PZ - Z^*P = -C^*(iJ)C, \]
which agrees with (2.7).

The space \( \mathfrak{M} \) can be used to provide a test to check whether a Nevanlinna pair \( (M(\tau),N(\tau)) \) is or is not a solution of the interpolation problem (IP). To formulate this test we associate with each ordered pair \( (M(\tau),N(\tau)) \) of matrix functions satisfying (2.14) the linear mapping \( \tau = \tau_{M,N} \) from the space \( \mathfrak{M} \) to some linear space of functions defined by
\[ (rF)(\tau) = (-M^2(\tau):N^2(\tau))F(\tau), \quad F \in \mathfrak{M}, \]
cf. (2.17). If \( \tau \) maps \( \mathfrak{M} \) into the space \( \mathcal{L}(M,N) \), then the same is true for each ordered pair equivalent to \( (M(\tau),N(\tau)) \) and we shall say that \( \tau \) associated with the Nevanlinna pair \( (M(\tau),N(\tau)) \) takes \( \mathfrak{M} \) into \( \mathcal{L}(M,N) \), cf. the remarks after Theorem 2.5. The test is stated in the first part of the next theorem. For the definition of the \( r \times r \) matrix \( P_{M,N} \), appearing in the second part of the theorem, we refer to the Introduction, formula (0.8).

**Theorem 3.1.** (i) The Nevanlinna pair \( (M(\tau),N(\tau)) \) is a solution of the interpolation problem (IP) if and only if the mapping \( \tau \) takes \( \mathfrak{M} \) into \( \mathcal{L}(M,N) \). (ii) If the Nevanlinna pair \( (M(\tau),N(\tau)) \) is a solution of the interpolation problem (IP), then
\[ [\tau F, \tau F]_{\mathcal{L}(M,N)} = P_{M,N}, \]
i.e., in terms of the matrix components \( [\tau F_{j0}, \tau F_{ip}]_{\mathcal{L}(M,N)} = (P_{M,N})^p_q \).

To prove the theorem we need the following technical lemma, which we also apply in the proof of Theorem 4.1 in Section 4.

**Lemma 3.2.** Let \( (M(\tau),N(\tau)) \) be an ordered pair of holomorphic matrix functions on \( \mathbb{C} \setminus \mathbb{R} \) that satisfies (2.14) and is a solution of the interpolation problem (IP). Then there exist unique vectors \( e_{jq} \in \mathbb{C}^n \), \( j = 1,2,\ldots,m \), \( q = 0,1,\ldots,r_j \), such that the coefficients \( c_{j0} \) of \( F_{jq}(\tau) \) in (3.1) are given by
\[ c_{j0} = \sum_{h=0}^{h-1} \left( (D_\tau)^{h+1} \left( M(\tau) \right)^{-1} N(\tau) \right) \mid_{\tau = \bar{\tau}, \tau = \tau_{j0}}, \quad j = 1,2,\ldots,m, \quad h = 0,1,\ldots,r_j. \]