Smooth points in some spaces of bounded operators

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We determine the smooth points of certain spaces of bounded operators \( L(X, Y) \), including the cases where \( X \) and \( Y \) are \( l^p \)- or \( c_0 \)-direct sums of finite dimensional Banach spaces or subspaces of the latter enjoying the metric compact approximation property. We also remark that the operators not attaining their norm are nowhere dense in \( L(X, Y) \) whenever \( K(X, Y) \) is an \( M \)-ideal in \( L(X, Y) \).

Let \( X \) be a Banach space and denote by \( B_X \) and \( S_X \) the unit ball and unit sphere of \( X \), respectively, and by \( K(X, Y) \) and \( L(X, Y) \) the spaces of compact and bounded linear operators from \( X \) into a second Banach space \( Y \). A point \( x \) of \( X \) is called smooth if the norm is differentiable at \( x \). Equivalently, \( x \) is smooth iff there is exactly one functional \( x^* \in S_X \) with \( \|x\| = x^*(x) \).

The aim of the present note is to prove the following extension of [9, Theorem 2] (unexplained notation will be defined below):

**Theorem 1** Suppose that \( X \) and \( Y \) are infinite dimensional and separable Banach spaces, which belong to classes \( (M_p) \) and \( (M_q) \) for some \( 1 < p, q \leq \infty \). Then \( T \) is a smooth point of the unit sphere of \( L(X, Y) \) if and only if

(a) The essential norm of \( T \), i.e. the number \( \|T\|_e = \inf_{K \in K(X)} \|T - K\| \), is strictly less than one.

(b) There is exactly one \( y_0^* \) in the unit ball of \( Y^* \) where \( \|T^*(y_0^*)\| = 1 \).

(c) The point \( T^*y_0^* \) is smooth.

The proof will be a direct consequence of the characterization of extreme points of \( B_{K(X, Y)} \) due to Rueß, Stegall, Lima and Olsen as well as of Lemma 6 and Lemma 7 below. Note that the finite dimensional case is already contained in [16] and [10].

Before we come to the proof of Theorem 1, let us give an example:
Put \( X = Y = c_0 \) and represent a norm one operator \( T \in L(c_0) \) by a matrix \((a_{ij})\). Since \( T^* \) attains its norm at an extreme point of \( B_{11} \), there is, for a smooth point \( T \), exactly one row \((a_{ij})_j\) with \( \sum_{j=1}^{\infty} |a_{ij}| = 1 \). Furthermore, condition (a) of the above statement is equivalent to the fact that there is \( \varepsilon > 0 \) with \( \sum_{j=1}^{\infty} |a_{ij}| < 1 - \varepsilon \) for all \( i \neq i_0 \). Finally, a point \((\lambda_n) \in l^1\) is smooth iff \( \lambda_n \neq 0 \) for all \( n \), so we have found:

**Corollary 2** A norm one operator \( T = (a_{ij}) \) on \( c_0 \) is a smooth point of the unit sphere, iff there is \( i_0 \in \mathbb{N} \) such that \( a_{i_0j} \neq 0 \) for all \( j \), \( \sum_{j=1}^{\infty} |a_{i_0j}| = 1 \), and, for some \( \varepsilon > 0 \), \( \sum_{j=1}^{\infty} |a_{ij}| < 1 - \varepsilon \) for all \( i \neq i_0 \).

Let us fix some further notation. A subspace \( J \) of \( X \) is called an \( M \)-ideal whenever there is \( J^* \) in \( X^* \) with
\[
X^* = J^\perp \oplus J^* ,
\]
where the expression \( X \oplus_p Y \) means that on the direct sum \( X \oplus Y \) the norm is given by \( \|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p} \) for \( 1 \leq p < \infty \) and by the sup-norm in the remaining case. The notation \( J^* \) is not accidental in the above: This space can (and always will) be identified with the dual of the \( M \)-ideal \( J \). We will make use of the following easy fact.

**Lemma 3** If \( J \) is an \( M \)-ideal in \( X \) then the set \( \text{ex } B_{X^*} \) of extreme points of \( B_{X^*} \) admits a decomposition
\[
\text{ex } B_{X^*} = \text{ex } B_{J^*} \cup \text{ex } B_{J^\perp}.
\]

The space \( X \) is said to belong to the class \((M_p)\) for \( 1 \leq p \leq \infty \), whenever \( K(X \oplus_p X) \) is an \( M \)-ideal in \( L(X \oplus_p X) \). Equivalently, these classes can be characterized by the asymptotic behaviour of nets \((K_\alpha)\) in \( B_{K(X)} \) converging strongly to the identity. For any of these spaces, \( K(X) \) is an \( M \)-ideal in \( L(X) \). The classes \((M_p)\) include for \( 1 < p < \infty \) the \( \ell^p \)-sums as well as (for \( p = \infty \)) the \( c_0 \)-sums of finite dimensional Banach spaces. They are also stable under passage to subspaces with the metric compact approximation property. The class \((M_1)\) consists of all finite dimensional Banach spaces.

For more information on these classes the reader is referred to [13] and [15]. Note that it is an open problem whether there is any exceptional space \( X \), not belonging to any of the classes \((M_p)\), for which \( K(X) \) is an \( M \)-ideal in \( L(X) \). What we will need to know about these spaces here, is contained in the following

**Theorem 4** (a) Suppose that the infinite dimensional Banach space \( X \) belongs to the class \((M_p)\) for \( 1 < p \leq \infty \). Then any sequence \((x_n)\) of normalized vectors converging weakly to zero has a subsequence equivalent to the standard basis of \( \ell^p \) (if \( p < \infty \)) or \( c_0 \) (if \( p = \infty \)). In addition, the closed linear span of the \( x_n \) is complemented in \( X \).

(b) If \( X \) is the dual of a space \( X_* \), the latter of which belongs to the class \((M_\infty)\), then any normalized sequence \((x_n)\) converging to zero in the weak*-topology has a subsequence equivalent to the unit vector basis of \( l^1 \). The closed linear span of this sequence is again complemented.