The basic sequence problem for quasi-normed spaces

By

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Introduction. It is an old and still unsolved problem whether every complete metric linear space (F-space) must have a proper closed infinite-dimensional subspace. This problem was raised in the sixties by Pełczyński (cf. [13]). We say that an F-space with no proper closed infinite-dimensional subspace is atomic. Some positive results are known. In [8], it is proved that if a F-space has strictly weaker Hausdorff vector topology (i.e. the space is not minimal) then it contains basic sequences. Hence it has a proper closed infinite-dimensional subspace. It is also shown that the F-space ω of all sequences is minimal and any minimal space that has a basis is isomorphic to ω. Since ω is not locally bounded, a minimal locally bounded F-space contains no basic sequences. We would like to point out that the existence of a minimal locally bounded F-space is not known. In 1983, Bastero's result [2] implies that no subspace of $L_p[0, 1]$ ($0 < p < 1$) can be minimal or atomic. There is a recent example due to Reese [15]. She constructed a nonlocally bounded F-space $X$ with a sequence of finite dimensional subspaces such that $\dim V_n \geq n$, and if $x_n \in V_n$ with $x_n \neq 0$ for infinitely many $n$ then $[x_n] = X$.

In this paper, we will show that an $A$-convex space (i.e. a space that has an equivalent plurisubharmonic quasi-norm) contains basic sequences by using the nonlinear method due to Ghoussoub and Maurey [6]. Since $L_p[0, 1]$ is $A$-convex, our result extends Bastero's result.

Definition. Throughout this paper all vector spaces are assumed to be complex. If $X$ is a vector space then a map $x \mapsto \|x\| (X \to \mathbb{R}_+)$ is called a quasi-norm if

i) $\|ax\| = |a| \|x\|$, for $a \in \mathbb{C}, x \in X$,

ii) $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$, $x_1, x_2 \in X$

where $C$ is a constant independent of $x_1, x_2$.

iii) if $\|x\| = 0$, then $x = 0$.

We call $(X, \|\cdot\|)$ a quasi-normed space. If a quasi-norm satisfies $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$ where $0 < p < 1$, then that quasi-norm is called a $p$-norm and $(X, \|\cdot\|)$ is called a $p$-normed space. It is known that every locally bounded F-space can be given

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a quasi-norm such that it induces the same topology on X. The converse is also true. Moreover, we can always assume it is a p-norm for some $0 < p < 1$ [12], [16].

Suppose $(X, \| \cdot \|)$ is a quasi-normed space. Let $\phi : X \to \mathbb{R}$ be a continuous function. $\phi$ is called plurisubharmonic if $\forall x_1, x_2 \in X$,

$$\phi(x_1) \leq \frac{2\pi}{\int_0^{2\pi} \phi(x_1 + e^{i\theta} x_2) \frac{d\theta}{2\pi}}.$$  

If $\| \cdot \|$ is plurisubharmonic, then $(X, \| \cdot \|)$ is A-convex. Etter [5] observed that $L_p[0, 1], 0 < p < 1$ is A-convex and Aleksandrov [1] showed that $L_p/H_p$ is not. In fact, there does not exist any nonzero plurisubharmonic quasi-norm on $L_p/H_p$. In [11] it is showed that every A-convex quasi-normed space can be given an equivalent plurisubharmonic p-norm $\| \cdot \|, 0 < p < 1$. Therefore, it is enough to consider A-convex spaces equipped with a plurisubharmonic p-norm in the rest of this paper.

**Basic sequences in A-convex p-normed spaces.** Before proving our theorem, we would like to mention some facts:

i) *(Dvoretzky's Theorem for the complex p-normed spaces)* [10], [4], [7] If $X$ is an infinite dimensional p-normed space, then there is a sequence of subspaces of $X$, $\{X_j\}$, such that

$$d(X_j, l_2) \to 1 \text{ as } j \to \infty.$$  

ii) For any $\alpha > 0$ and a non-negative integer $k$, there is $\rho(k, \alpha) > 0$ such that if $S$ is the sphere of the $n$-dimensional euclidean space $n \leq k$ with the normalized Lebesgue measure $\mu$ and $A \subset S$ such that $\mu(S - A) < \rho(k, \alpha)$, there exists a complex space $Y$ of dimension $k$ so that $\forall x \in Y \cap S$, dist $(x, A) < \alpha$.

iii) Suppose $\phi : S \to \mathbb{R}$ is a p-Lip function, $0 < p < 1$, i.e. there is a constant $C > 0$ such that $|\phi(x) - \phi(y)| \leq C \| x - y \|_p \forall x, y \in S$. We define $\| \phi \|_{\text{Lip}, p}$ to be the smallest $C$ in the inequality. If $m$ is the median of $\phi$ on $(S, \mu)$ and $\| \phi \|_{\text{Lip}, p} \leq M$, then $\forall x > 0$,

$$\mu \{ x \in S | |\phi(x) - m| > M x^\alpha \} \leq 2 \exp(-n x^2).$$

The reader is referred to [14] for more details.

**Lemma 1.** Let $X$ be a p-normed space and $(\varepsilon_j)_{j=1}^\infty$ a sequence of positive reals that tends to 0. For every $N$-tuple $\beta = (\phi_1, \phi_2, \ldots, \phi_N)$ of p-Lip functions on $X$ such that $\| \phi_j \|_{\text{Lip}, p} \leq M$ for every $j = 1, \ldots, N$, there exists a subspace $Y_0$ with $d(Y_0, l_2^N) \leq 1 + \varepsilon_N$ such that if $S_\beta = S_{Y_0}$, then for every $j = 1, \ldots, N$,

$$\text{osc}(\phi_j, S_\beta) \leq M \varepsilon_N^\alpha.$$  

**Proof.** Use i) to choose $n$ large enough so that

$$d(X_n, l_2^N) \leq 1 + 4^{-\frac{1}{p}} \varepsilon_N$$

for some subspace $X_n$ of $X$ and

$$2N \exp(-n 8^{-\frac{2}{p}} \varepsilon_N^2) \leq \rho(N, 8^{-\frac{1}{p}} \varepsilon_N).$$

Since $d(X_n, l_2^N) \leq 1 + 4^{-\frac{1}{p}} \varepsilon_N$, we can find an Euclidean norm $\| \cdot \|_2$ on $X_n$ such that $\| x \| \leq \| x \|_2 \leq (1 + 4^{-\frac{1}{p}} \varepsilon_N) \| x \| \forall x \in X_n$. 
