ON SELF-DUAL SUBNORMAL OPERATORS

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We improve the result of C. C. Huang about self-dual subnormal operators, and consider the converse of this result.

1. INTRODUCTION

Let $H$ be a Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on $H$. Then $S \in B(H)$ is said to be subnormal if there exist a Hilbert space $K$ containing $H$ and a normal operator $N \in B(K)$ such that $Sf = Nf$ for every $f \in H$. Such an $N$ is called a normal extension of $S$. Every normal operator is obviously subnormal. And, if $S$ is subnormal, then $S$ is hyponormal, i.e., $S^*S - SS^* \geq 0$. An operator $Q \in B(H)$ is said to be quasinormal if $(Q^*Q - QQ^*)Q = 0$. It is known that every quasinormal operator is subnormal. If $S \in B(H)$ is subnormal and $N \in B(K)$ is a normal extension of $S$, then $N$ is called a minimal normal extension of $S$ if $K$ has no proper subspace that contains $H$ and reduces $N$. If $S$ is subnormal, then a minimal normal extension of $S$ is unique up to unitary equivalence, therefore it is called the minimal normal extension of $S$. For these facts, see [2].

Let $S \in B(H)$ be a subnormal operator, and let $N \in B(K)$ be its normal extension. With respect to the decomposition $K = H \oplus H^\perp$, $N$ can be written as a $2 \times 2$ matrix with operator entries:

$$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix}. \quad (1)$$

If the decomposition $K = H^\perp \oplus H$ is considered, then

$$N^* = \begin{pmatrix} T & X^* \\ 0 & S^* \end{pmatrix}.$$

From this it is clear that $T$ is subnormal and $N^*$ is a normal extension of $T$. An operator $A \in B(H)$ is said to be pure if $A$ has no nonzero reducing subspace on which it is normal. R. F. Olin has observed that $S$ is pure if and only if $N^*$ is the minimal normal extension of $T$. (See [2]. Although in [2] $N$ is assumed to be the minimal normal extension of $S$, the condition that $N$ is minimal is not necessary in the proof of Olin's theorem in [2].) Now let us recall the definition of the dual of a pure subnormal operator, which was introduced by J. B. Conway [1].
DEFINITION. If $S$ is a pure subnormal operator and $N$, the minimal normal extension of $S$, has the representation (1), then the operator $T$ in (1) is called the dual of $S$.

Notice that $T$ is unique up to unitary equivalence. J. B. Conway [1] showed that if $S$ is pure subnormal and $T$ is the dual of $S$, then $S$ is the dual of $T$ and $\sigma(S) = \{\lambda; \lambda \in \sigma(T)\}$, where $\sigma(T)$ denotes the spectrum of $T$. He also introduced the following definition of a self-dual subnormal operator.

DEFINITION. A pure subnormal operator is said to be self-dual if it is unitarily equivalent to its dual.

Every pure quasinormal operator is a self-dual subnormal operator (See [1]). In this paper we improve the result of C. C. Huang [3] about self-dual subnormal operators.

2. AN EXAMPLE OF A SELF-DUAL SUBNORMAL OPERATOR

The following lemma was proved by G. J. Murphy [5]. By using Olin's theorem, we give a different proof.

LEMMA 1. Let $S \in B(H)$ be a pure operator. Then, $S$ is a self-dual subnormal operator if and only if there exists an operator $X \in B(H)$ such that the operator matrix

$$
\begin{pmatrix}
S & X \\
0 & S^* 
\end{pmatrix}
$$
on $H \oplus H$ is normal.

PROOF. Suppose $S$ is a self-dual subnormal operator. Let $N = \begin{pmatrix} S & Y \\ 0 & T^* \end{pmatrix}$ on $K = H \oplus H^\perp$ be the minimal normal extension of $S$ such that $S$ is unitarily equivalent to $T$. If $U$ is an isomorphism from $H$ to $H^\perp$ such that $S = U^*TU$, then $V = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$ is an isomorphism from $H \oplus H$ to $K = H \oplus H^\perp$, and

$$
V^*NV = \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} S & Y \\ 0 & T^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} S & YU \\ 0 & S^* \end{pmatrix}.
$$

Hence, $\begin{pmatrix} S & YU \\ 0 & S^* \end{pmatrix}$ on $H \oplus H$ is normal.

To prove the sufficiency, it is enough to show that $\begin{pmatrix} S & X \\ 0 & S^* \end{pmatrix}$ is the minimal normal extension of $S$. An easy matrix calculation shows that $\begin{pmatrix} S & X^* \\ 0 & S^* \end{pmatrix}$ is also a normal extension of $S$. Thus, since $S$ is pure, $\begin{pmatrix} S & X \\ 0 & S^* \end{pmatrix}$ is the minimal normal extension of $S$ by Olin's theorem. 

Let $\{A\}'$ denote the commutant of $A$. For a normal operator $A$, $\{A\}' = \{A^*\}'$ holds by the Fuglede-Putnam Theorem. Considering this, we can construct the following example of a self-dual subnormal operator.