SECOND ORDER PARALLEL ALGORITHMS FOR FREDHOLM INTEGRAL EQUATIONS WITH CONTINUOUS DISPLACEMENT KERNELS

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Parallel algorithms with second order precision are derived for efficient numerical solution of Fredholm integral equations of the second kind with displacement kernels.

Introduction
In the present paper we study efficient methods of solution of Fredholm integral equations of the second kind with a displacement kernel. That is, equations of the form

\[ h(t) - \int_0^T k(t-r)h(r)dr = f(t), \quad 0 \leq t \leq T \quad (0.1) \]

where \( k(t) \) is a \( p \times p \) matrix-valued function. Throughout the paper we assume that \( k(t) \) has (at least) one bounded derivative. We also assume that the family of equations

\[ h(t,\xi) - \int_0^\xi k(t-r)h(r,\xi)dr = f(t), \quad 0 \leq t \leq \xi \leq T \quad (0.2) \]

has a unique continuous solution for each \( \xi \in [0,T] \) and each continuous right-hand side \( f(t) \). This is the case when the kernel \( k(t-s) \) is positive definite for example.

The main purpose of this paper is to use properties of the kernel \( k(t-s) \), such as its displacement structure and smoothness to obtain efficient second-order algorithms for the numerical solution of these equations. In particular, we are interested in parallel algorithms for the solution of (0.1) that
take advantage of the displacement structure of the kernel and
the existence of bounded second derivatives.

Our analysis is based on the results obtained by
Gohberg and Koltracht ([5]) for integral equations and on
discrete analogues of these results by Gohberg, Kailath,
Koltracht and Lancaster ([4]).

Algorithms for the solution of (0.1) are obtained in
two ways. One method is based on the fact that the solutions of
the equations
\[
\begin{align*}
P(t,\xi) - \int_0^t k(t-r)p(r,\xi)dr &= k(t) \\
P(t,\xi) - \int_0^\xi k(r-t)p(r,\xi)dr &= k(-t)
\end{align*}
\]
also solve the initial value problem
\[
\begin{align*}
\frac{\partial}{\partial \xi} P(t,\xi) &= P(t-t,\xi)P(\xi,\xi) & &0 \leq \xi \leq T \\
\frac{\partial}{\partial t} P(t,\xi) &= P(t-t,\xi)P(\xi,\xi) & &\xi-T \leq t \leq T \\
P(t,0) &= k(t), & P(t,0) &= k(-t), & 0 \leq t \leq T.
\end{align*}
\]
This problem is then solved by a second order Heun's difference
scheme of Runge-Kutta type.

The second method is based on an appropriate
discretization of the equation (0.1) using the trapezoidal
quadrature rule. This leads to a linear system of equations with
a coefficient matrix that is a low rank perturbation of a
Toeplitz matrix. Parallel algorithms of [4] can then be used to
solve such a system.

The method based on the initial value problem (0.3) may
have the following advantage. In a search for an optimal
numerical method for solving (0.3) one does not have to bother
about the displacement structure of the kernel, which is already
used in the derivation of this initial value problem. In
contrast, an optimal quadrature method for solving (0.1) will
lead to a perturbed Toeplitz system of linear equations (as an