THEORY OF A LAVAL NOZZLE FOR A TWO-PHASE MIXTURE CONTAINING PARTICLES OF SMALL LAG

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A two-velocity and two-temperature model is considered for a continuous medium in relation to the flow of a mixture of gas and particles in the subsonic, transsonic, and supersonic parts of a Laval nozzle. It is assumed that the particles are small, and hence that the coefficients $\phi^f$ and $\phi^q$, which define the interaction with the gas, are large (these coefficients are inversely proportional to the square of the particle radius for a Stokes mode of flow). This means that the velocity or thermal lag of the particles relative to the gas is small. The solution is sought as expansions with respect to the small parameters $\varepsilon_1 = 1/\phi^f$ and $\varepsilon_2 = 1/\phi^q$.

The first problem is one of special perturbations, which arise from the formation of a layer of pure gas near the wall on account of particle lag; if $\varepsilon_1 \rightarrow 0$, the thickness of this layer tends to zero, but the difference in the values of the gas parameter at the wall and at the boundary of the layer remains finite. Equations are derived that describe with accuracies $\varepsilon_1$ and $\varepsilon_2$ the flow of the mixture of gas and particles at the core, together with equations that define the gas parameters in the wall layer. It is found that this layer resembles an ordinary boundary layer in that the pressure change across it is a quantity of higher order than the change in the other parameters. To solve the equations that define the characteristics of the mixture at the core, use is made of an expansion with respect to the small parameter $\varepsilon_1 = 1/R$, where $R$ is the radius of curvature of the nozzle wall in the minimal cross section, as referred to the radius at that section.

The two-phase flow in a Laval nozzle has been considered [1-3] in the one-dimensional approximation by expansion with respect to $\varepsilon_1$ and $\varepsilon_2$ for one of these parameters; the expansion with respect to $\varepsilon_1$ used here is analogous to the method of [4] for solving problems in the theory of Laval nozzles for pure gas. Exact published results are available within the framework of the two-liquid model for the flow of a mixture of gas and particles, but these are restricted to the supersonic part of the nozzle, where the parameters of the gas and particles are derived by the direct methods of characteristics [5-11] or by inverse methods [12-14].

On the other hand, we are not aware of any papers in which solutions have been obtained for the theory of Laval nozzles for nonequilibrium two-phase flow for the supersonic and subsonic or transsonic parts without involving additional assumptions; of the available approximate approaches, we may note the very widely used method of integrating the equations of energy and motion for the particles (these equations are ones in total derivatives along the particle flow lines), which are used for the equilibrium parameters of the mixture [15], while there is also the method of [16], in which it was assumed that the distributions of the pressure and the inclination of the gas velocity vector are the same for the equilibrium and nonequilibrium two-phase flows. There is some justification for using these approximate approaches for comparatively small relative flow rates of the particles, but at high relative flow rates the assumptions may involve substantial errors.

The presence of a layer of pure gas near the wall also hinders the formulation and solution of the inverse problem, while the method of [17-19], which is very effective for a pure gas, becomes difficult to use when there is little particle lag and the layer is thin. It may be that the relationships derived below
for the wall layer should be used in conjunction with the methods noted above. In this connection we must
stress that these relationships apply for any \( \epsilon \) when the layer is thin by comparison with the characteristic
dimension of the nozzle. This arises because \( \epsilon \) appears in the equations only via the thickness of the layer
when there are no particles, and this thickness, as we shall see, is proportional to \( \epsilon \) when the particle lag
is small.

1. We use a rectangular or cylindrical system of coordinates to consider the flow of gas mixed with
particles in a planar or axially symmetrical Laval nozzle (Fig. 1). We locate the origin in the plane
of minimum cross section of the nozzle, with the \( x \) axis directed from left to right (along the flow) along
the axis or symmetry plane, while the \( y \) axis is perpendicular to the \( x \) axis. If there are no external
sources of heat or force, and if we neglect the volume of the particles, we get the following equations for the flow
within the framework of the two-fluid model [5-14, 20]:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\rho}{\rho} \frac{\partial f}{\partial x} (u - u_s) &= 0 \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\rho}{\rho} \frac{\partial f}{\partial y} (v - v_s) &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\rho}{\rho} \frac{\partial f}{\partial x} (u - u_s) + \frac{\rho}{\rho} \frac{\partial f}{\partial y} (v - v_s) &= 0
\end{align*}
\]

(1.1)

Here \( p \) is pressure, \( h \) is specific enthalpy, \( T \) is temperature, \( \rho \) is density, \( u \) and \( v \) are the projections
of the gas velocity vector on the \( x \) and \( y \) axes, \( T_s, \rho_s, u_s, \) and \( v_s \) are the corresponding quantities for the
particles, \( \epsilon_s \) is the specific internal energy of the particles, and the coefficients \( \psi^f \) and \( \psi^q \), which we take
to be constants, which corresponds to Stokes flow around each particle, characterize the dynamic and therm
al interaction between the gas and the particles; \( \nu = 0 \) or \( 1 \) respectively in the planar and axially symmetrical
cases. We assume that the gas is perfect, with constant specific heats and ratio \( \kappa \), while the internal
energy of the particles is a linear function of the temperature (\( \delta \) is a constant equal to the specific heat of
the particles).

All the quantities are taken as dimensionless in (1.1) and subsequently. Let \( L, q_s, \) and \( \rho_s \) be character-
istic quantities with the dimensions of length, velocity, and density, while \( R \) is the dimensional value of the
gas constant. Then we perform the reduction to dimensionless form by referring the spatial variables
to \( L \), the velocities to \( q_s \), the densities to \( \rho_s \), the pressures to \( \rho_s q_s^2 \), the enthalpy and internal energy to \( q_s^2 \),
the temperature to \( q_s^2/R \), and the specific heat of the particles to \( R \). As \( L \) we take the radius or half height
of the minimal section of the nozzle, while as \( q_s \) and \( \rho_s \) we take the critical velocity and density for the
equilibrium flow, i.e., flow without particle lag, where \( u_s = u, v_s = v, \) and \( T_s = T \).

The boundary conditions for (1.1) are the condition for absence of flow at the wall, which is specified
by the equation \( y = y_w(x) \), and the symmetry condition for \( y = 0 \), these taking the form

\[
v(x, y_w) = y_w'(x) u(x, y), \quad v(x, 0) = 0
\]

(1.2)

and

\[
u_s = u, \quad v_s = v, \quad T_s = T \quad (x \to -\infty)
\]

(1.3)

The prime in (1.2) and subsequently denotes the total derivative with
respect to the corresponding argument (in this case \( x \)).

Equation (1.3) applies when the nozzle is joined on the left to a semi-
infinite cylindrical tube (here the vertical components of the velocities of
gas and particles tend to zero, while the limiting values of the horizontal
components differ from zero) and when the nozzle expands without limit
for \( x \to -\infty \) (then both components of the velocities tend to zero). Apart