Factorization of vector measures

By

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Motivated by recent results of Schaefer and Zhang [10], [11] we will show that a uniformly countably additive (strongly additive) family of vector measures can be characterized by a simultaneous factorization through a Banach lattice $F$ with order continuous norm via a single $\sigma$-order continuous operator (positive operator respectively) into $F$. This factorization (for a single operator) first was established in [6] for a (positive) weakly compact operator on a $C(K)$-space. Several years later this factorization method was reproved for an order-weakly compact operator by Niculescu [8] and by Ghoussoub and Johnson [3] for an operator into a Banach space not containing a subspace isomorphic to $c_0$. This present paper mainly is concerned with countably additive vector measures ($\sigma$-order norm continuous operators). Once established, the factorization theorems can be used together with several results from the general theory of positive operators on Riesz spaces to get simple proofs of almost all known results concerning the structure of vector measures, see [5] for details. Moreover, therein many results of this present paper can be found.

We will use [1] as a general reference for vector measures and [7], [9] as general references for Riesz spaces, Banach lattices and linear operators defined on the spaces.

Throughout this paper let $E, F$ be Archimedean Riesz spaces and $X, Y$ Banach spaces. Moreover, we suppose that $E$ is $\sigma$-Dedekind complete. Recall that a linear operator $T: E \to X$ is said to be order-weakly compact if $T[0, x] \subseteq X$ is relative weakly compact for all $x \in E_+$. Let $(\Omega, \Sigma)$ be a measurable space. Since $\Sigma$ is a $\sigma$-algebra, it is easy to see that $B(\Sigma)$, the space of all bounded measurable functions on $\Omega$, is a $\sigma$-Dedekind complete $M$-space. Moreover, every vector measure $\mu: \Sigma \to X$ of bounded semivariation uniquely extends to an operator $T_\mu: B(\Sigma) \to X$. It is well-known that $\mu$ is strongly additive, iff $T_\mu$ is order-weakly compact, or equivalently in this situation that $T_\mu$ is weakly compact, or equivalently in this situation that $T_\mu$ is order-weakly compact. Compare [7], Sect. 3.4 and 3.9 or [1].

1. Definition. A linear operator $T: E \to X$ is said to be $\sigma$-order-norm continuous if $\|Tx_n\| \to 0$ as $n \to \infty$ for every sequence $(x_n)_n \subseteq E$ satisfying $x_n \downarrow 0$ as $n \to \infty$.

It is easy to see that every $\sigma$-order-norm continuous operator is order-weakly compact (provided $E$ is $\sigma$-Dedekind complete).

2. Proposition. A vector measure of bounded semivariation $\mu: \Sigma \to X$ is countably additive if and only if $T_\mu: B(\Sigma) \to X$ is $\sigma$-order-norm continuous.
Proof. If $T_\mu$ is $\sigma$-order-norm continuous, then it is easy to see that $\mu$ is countably additive. To prove the converse, assume that $B(\Sigma) \ni f_n \downarrow 0$ as $n \to \infty$. Let $\varepsilon > 0$ and $A_n = \{ f_n \geq \varepsilon \}$. From $A_n \downarrow \phi$ as $n \to \infty$ we conclude that $\mu(A_n) \to 0$ as $n \to \infty$. Consequently,

$$\limsup_{n \to \infty} \| T_\mu f_n \| \leq \varepsilon M,$$

where $M$ is the norm of the semivariation of $\mu$. □

To make the situation more transparent we assume throughout this paper that the Riesz space $E$ is $\sigma$-Dedekind complete, although several results hold for more general classes of Riesz spaces, in particular those results concerning order-weakly compact operators, compare [7], Sect. 3.4.

A subset $A \subset E_-$ is said to be locally $\sigma(E_-, E)$-sequentially precompact if the restriction of $A$ to $E_u$ is $\sigma(E_u^+, E_u)$-sequentially precompact for all $u \in E_+$. Of course, every $\sigma(E_-, E)$-sequentially precompact subset $A \subset E_-$ also is $\sigma(E_-, E)$ sequentially precompact. The converse, however, need not be true. Moreover, if $T: E \to X$ is order weakly compact, then $A = T^\sim$ ball $(X')$ is locally $\sigma(E_-, E)$-sequentially precompact; in this case it follows that $\rho_A(x) = q_T(x)$ for all $x \in E$ where

$$\rho_A(x) = \sup \{ \langle |x'|, |x| \rangle | x' \in A \}$$

and

$$q_T(x) = \sup \{ \| T_n \| | y | \leq | x | \};$$

see [7], 3.4.3. The following proposition is a sharpened version in a slightly different form of the factorization theorem [7] 3.4.6 for order-weakly compact operators. This present form, in addition, is concerned with order continuity (countably additivity).

3. Proposition. Assume that $\phi \equiv A \subset E_-$ is locally $\sigma(E_-, E)$-sequentially precompact, and let $\rho = \rho_A$ be seminorm generated by $A$:

$$\rho(x) = \sup \{ \langle |x'|, |x| \rangle | x' \in A \}.$$

$F_0 = (E/\rho^{-1}(0), \rho)$ is a normed Riesz spaces. The completion $F$ of $F_0$ is a Banach lattice with order continuous norm and the quotient map $Q: E \to F_0$ considered as an operator $Q: E \to F$ is an almost interval preserving lattice homomorphism.

Moreover, if in addition $A \subset E^-_\sim$, then $Q$ is $\sigma$-order continuous.

Proof. $\rho$ is a lattice seminorm, hence $\rho^{-1}(0)$ is an ideal and the quotient map $Q: E \to F_0 = E/\rho^{-1}(0)$ is an interval preserving lattice homomorphism [7], 1.3.13 and 1.4.18. Since the restriction of $A$ to $E_u$ is $\sigma(E_u^+, E_u)$-sequentially precompact for all $u \in E_+$, it follows from [7], 2.5.1 that

$$\| Q x_n \| = \rho(x_n) \to 0 \quad \text{as} \quad n \to \infty$$

for every order bounded disjoint sequence $(x_n)_n \subset E_+$. By [7], 3.4.4, $Q$ is order-weakly compact. Consequently $[0, Q x] \subset F$ is weakly compact for all $x \in E_+$. A simple density argument shows that $[0, y]$ is weakly compact for all $y \in F_+$. [7], 2.4.2 implies that the norm on $F$ is order continuous.