We consider the fundamental solutions of a wide class of first order systems with polynomial dependence on the spectral parameter and rational matrix potentials. Such matrix potentials are rational solutions of a large class of integrable nonlinear equations, which play an important role in different mathematical physics problems. The concept of bispectrality, which was originally introduced by Grünbaum, is extended in a natural way for the systems under consideration and their bispectrality is derived via the representation of the fundamental solutions. This bispectrality is preserved under the flows of the corresponding integrable nonlinear equations. For the case of Dirac type (canonical) systems the complete characterization of the bispectral potentials under consideration is obtained in terms of the system's spectral function.

1 Introduction

The striking characteristic of certain aspects of operator theory is its deep connections with other apparently distant mathematical fields. An example of such connections is provided by time and band limiting operators [BGK97], [GKW93], [Sle83]. The present work has one of its roots in the works of D. Slepian and collaborators motivated by applications to communication theory [SP61], [Sle64], [Sle78]. The simplest instance of the problem that was investigated by D. Slepian concerns the problem of determining in an effective way the eigenvalues and eigenfunctions of the time and band limiting operator:

$$f(x) \rightarrow \int_{-T}^{T} \frac{\sin(\Omega(x-s))}{x-s} f(s) \, ds.$$  

Here, $2T$ is the size of the time interval and $2\Omega$ is the size of the frequency interval. The question resurfaced in extensions developed by A. Grünbaum stimulated by problems in computerized tomography. For a historical description of this problem the reader is referred to [Sle83], [Grü82], [Zub97] and references therein. The time and band limiting problem turned out to have fundamental connections with the main object of the present article, which is the so-called bispectral problem [Grü84], [Grü92], [Grü94]. The latter goes as follows:

When does an ordinary differential operator $L(x, \partial_x)$ admit a family of eigenfunctions $\varphi(x, \lambda)$ that also satisfies a differential equation in the spectral parameter $\lambda$ of the
where $\Theta(x)$ is independent of $\lambda$ and $B$ is a positive order ordinary differential operator independent of $x$. In other words, $\varphi$ satisfies both (1.1) and

$$L(x, \partial_x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda).$$  

(1.2)

The presence of two spectral problems, one with spectral variable $\lambda$ and the other with (bi)spectral variable $\Theta(x)$, leads to the terminology \textit{bispectral problem}.

The bispectral problem for the scalar Schrödinger operator case $L = -\partial_x^2 + u$ was completely characterized by Duistermaat and Grünbaum [DG86]. One of the directions towards which the bispectral property was explored was that of systems, such as those studied by Zakharov-Shabat [ZS71] and Ablowitz-Kaup-Newel-Segur [AKNS74] as well as more general matrix differential operators. This was pursued, for example, in [Zub90], [Zub92b], [BHY96].

The bispectral problem presented very surprising connections with the field of integrable systems and soliton theory. Besides some elementary examples, it turned out that the class of bispectral Schrödinger potentials included the \textit{rational solutions} of the Korteweg-de Vries (KdV) hierarchy [AS81], [AMM77], [AM78], [CC77], which play an important role in soliton theory. Such rational functions are limiting cases of $N$-soliton potentials, which in turn are limiting cases of algebraic-geometric solutions of the Kadomtsev-Petviashvili hierarchy [NMPZ84]. The connections between soliton theory and the bispectral problem did not stop here. Besides the rational solutions of the KdV, the remaining bispectral Schrödinger potentials showed an even deeper connection with infinite dimensional integrable systems. In a nutshell, they turned out to be rational solutions of the master symmetries of the KdV hierarchy. See [ZM91], [ZV00].

Although the scalar bispectral differential operators have been thoroughly investigated, the picture for matrix rational coefficients seems to be much less understood. See [HK98] and references therein for a recent account.

In this paper we shall consider systems with polynomial dependence on the spectral parameter $\lambda$:

$$\partial_x w(x, \lambda) = G(x, \lambda)w(x, \lambda), \quad G(x, \lambda) = -\sum_{k=0}^{r} \lambda^k q_k(x).$$  

(1.3)

The function $w$ in (1.3) is an absolutely continuous $m \times p$ matrix valued function. It can be either a fundamental solution or a vector function, in particular. Many systems auxiliary to important integrable nonlinear equations have this form. For systems of the form (1.3), we shall present large classes of rational coefficients where different forms of bispectrality manifest itself and construct the corresponding fundamental solutions of the form $\tilde{w}(x, \lambda) = w_A(x, \lambda)e^{ixN^d}$, where $d$ is a diagonal matrix and $w_A$ is rational both in $x$ and $\lambda$. After taking into account the additional time variable $t$, we show that such rational potentials satisfy different integrable nonlinear equations.

The main objective of this paper is to explore the bispectral property of the system (1.3) in case $r = 1$ and to generalize it in a natural way for $r > 1$. Another goal is to get a