TRACE CLASS NORM INEQUALITIES FOR LOCALIZATION OPERATORS

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We give sharp estimates on the norms in the trace class of localization operators in terms of their symbols.

1 The Main Results

Let $G$ be a locally compact and Hausdorff group on which the left Haar measure is denoted by $\mu$. Let $X$ be a separable and complex Hilbert space of which the dimension is infinite. We denote the inner product and the norm in $X$ by $(\cdot, \cdot)$ and $\| \cdot \|$ respectively. Let $B(X)$ be the $C^*$-algebra of all bounded linear operators on $X$ and let $\| \cdot \|$ denote the norm in $B(X)$. An irreducible and unitary representation $\pi : G \rightarrow B(X)$ of $G$ on $X$ is said to be square-integrable if there exists a nonzero element $\varphi$ in $X$ such that

$$\int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) < \infty. \quad (1.1)$$

We call any element $\varphi$ in $X$ for which $\| \varphi \| = 1$ and (1.1) is valid an admissible wavelet for the square-integrable representation $\pi : G \rightarrow B(X)$, and we define the constant $c_\varphi$ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g).$$

**Theorem 1.1** Let $\varphi$ be an admissible wavelet for the square-integrable representation $\pi : G \rightarrow B(X)$. Then

$$\langle x, y \rangle = \frac{1}{c_\varphi} \int_G \langle x, \pi(g)\varphi \rangle (\pi(g)\varphi, y) d\mu(g) \quad (1.2)$$

for all $x$ and $y$ in $X$.

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Remark 1.2 The formula (1.2) is known as the resolution of the identity formula. Theorem 1.1 is a simplified version of Theorem 3.1 in the paper [4] by Grossmann, Morlet and Paul. A proof can also be found in Chapter 1 of the book [6] by Wong.

Let $F \in L^1(G)$. Then for any $x$ in $X$, we define $L_F x$ to be the element in $X$ such that

$$
(L_F x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y) d\mu(g)
$$

(1.3)

for all $y$ in $X$. Then we have the following proposition, which is Proposition 2.1 in the paper [5] by He and Wong. See also Proposition 2.1 in the book [6] by Wong.

**Proposition 1.3** $L_F : X \to X$ is a bounded linear operator and

$$
\|L_F\| \leq \frac{1}{c_\varphi}\|F\|_{L^1(G)}.
$$

Remark 1.4 The bounded linear operator $L_F : X \to X$ is called the localization operator corresponding to the symbol $F$. The impetus for the terminology stems from the simple observation that if the symbol $F$ in $L^1(G)$ is replaced by the function $H : G \to \mathbb{C}$ given by

$$
H(g) = 1, \quad g \in G,
$$

then the resolution of the identity formula implies that the localization operator $L_H : X \to X$ is simply the identity operator on $X$. Thus, in general, the symbol $F$ is there to localize on $G$ so as to produce a nontrivial bounded linear operator on $X$ with various applications in the mathematical sciences. Localization operators studied in this paper are generalizations of localization operators studied in the paper [1] by Daubechies. See also Section 2.8 of the book [2] by Daubechies in this connection.

Let $A : X \to X$ be a compact operator. If we denote by $A^* : X \to X$ the adjoint of $A : X \to X$, then the linear operator $(A^*A)^{1/2} : X \to X$ is positive and compact. Let $\{\varphi_k : k = 1, 2, \ldots\}$ be an orthonormal basis for $X$ consisting of eigenvectors of $(A^*A)^{1/2} : X \to X$ and let $s_k(A)$ be the eigenvalue of $(A^*A)^{1/2} : X \to X$ corresponding to the eigenvector $\varphi_k, k = 1, 2, \ldots$. We say that the compact operator $A : X \to X$ is in the trace class $S_1$ if

$$
\sum_{k=1}^\infty s_k(A) < \infty,
$$

and we call $s_k(A), k = 1, 2, \ldots$, the singular values of $A$. It can be shown that $S_1$ is a Banach space in which the norm $\|\|_{S_1}$ is given by

$$
\|A\|_{S_1} = \sum_{k=1}^\infty s_k(A), \quad A \in S_1.
$$

(1.4)

If $A : X \to X$ is a linear operator in $S_1$ and $\{\varphi_k : k = 1, 2, \ldots\}$ is any orthonormal basis for $X$, then it can be proved that the series $\sum_{k=1}^\infty (A\varphi_k, \varphi_k)$ is absolutely convergent.