d-Cube Decompositions of $K_n \setminus K_m$

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Abstract. Necessary conditions on $n$, $m$ and $d$ are given for the existence of an edge-disjoint decomposition of $K_n \setminus K_m$ into copies of the graph of a $d$-dimensional cube. Sufficiency is shown when $d = 3$ and, in some cases, when $d = 2$. We settle the problem of embedding 3-cube decompositions of $K_m$ into 3-cube decompositions of $K_n$, where $n \geq m$.

1. Introduction

We denote the sets of vertices and edges of a graph $G$ by $V(G)$ and $E(G)$, respectively. For $m \leq n$, let $K_n \setminus K_m$ denote the subgraph of $K_n$ induced by $E(K_n) \setminus E(K_m)$; call $K_n \setminus K_m$ a complete graph of order $n$ with a hole of size $m$. Let $K'_x$ be the complete $r$-partite graph with exactly $x$ vertices in each part. Undefined graph theoretical terminology can be found in [1].

The $d$-cube is the graph $Q_d$ whose vertex set is the set of all binary $d$-tuples, and whose edge set consists of all pairs of vertices which differ in exactly one coordinate. It is easy to see that $Q_d$ has $2^d$ vertices, $d2^{d-1}$ edges and is $d$-regular and bipartite.

Let $G$, $H$ and $H'$ be graphs with $G \subseteq H \subseteq H'$. A $G$-decomposition of $H$ is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edge-disjoint subgraphs of $H$, each of which is isomorphic to $G$, such that the edge sets of the $G_i$'s partition the edge set of $H$. In this case, we say $G$ divides $H$ and write $G \mid H$. If $\Gamma'$ is a $G$-decomposition of $H'$ such that $\Gamma \subseteq \Gamma'$, then $\Gamma$ is said to be embedded in $\Gamma'$.

The decomposition of graphs, and embeddings of graph decompositions, have been and remain the focus of a great deal of research (see [2] for a thorough discussion of the subject of graph decompositions). In particular, $K_k$-decompositions of $K_n$ (see [2]) and $C$-decompositions of $K_n$, where $C$ is a cycle of given length [11],

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have received much attention. For an excellent reference on cycle decompositions and embeddings of cycle decompositions, the reader is directed to [11].

$G$-decompositions of $K_n \setminus K_m$ can lead to results on embeddings of graph decompositions. Some results on the decomposition of $K_n \setminus K_m$ into $k$-cycles can be found in [9] (when $k$ and $m$ are both odd), [5] (when $k$ is odd and $m$ is even) and [3] (when $k \leq 14$). $K_k$-decompositions of $K_n \setminus K_m$ are investigated in [8].

The decomposition of $K_n$ into $d$-cubes was first studied by Kotzig [10]. Several results on the decomposition of $K_m,n$ into $d$-cubes are presented in [6, 7, 13]. The decomposition of complete graphs and of complete bipartite graphs into 3-cubes and into 4-cubes is investigated in [4].

In this paper, we give necessary conditions (on $n$, $m$ and $d$) for the existence of a $Q_d$-decomposition of $K_n \setminus K_m$. We prove that these conditions are sufficient in the case $d = 3$ and in some cases when $d = 2^t$. We also show some embedding results.

In 1981, Kotzig [10] proved the following three results concerning $Q_d$-decompositions of $K_n$:

**Theorem 1.1.** If $d$ is even and there is a $Q_d$-decomposition of $K_n$, then $n \equiv 1 \pmod{d2^d}$.

**Theorem 1.2.** If $d$ is odd and there is a $Q_d$-decomposition of $K_n$, then either

(a) $n \equiv 1 \pmod{d2^d}$ or
(b) $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$.

**Theorem 1.3.** There is a $Q_d$-decomposition of $K_n$ if $n \equiv 1 \pmod{d2^d}$.

We will need the following two theorems from [4] (Theorem 1.4 first appeared in [12].)

**Theorem 1.4.** There exists a 3-cube decomposition of $K_n$ if and only if $n \equiv 1$ or $16 \pmod{24}$.

**Theorem 1.5.** For $m \leq n$, a 3-cube decomposition of $K_{m,n}$ exists if and only if $m \equiv n \equiv 0 \pmod{3}$, $mn \equiv 0 \pmod{4}$ and $m \geq 4$.

We will also make use of the following two theorems. A proof of Theorem 1.6 can be found in [6], and Theorem 1.7 is proved in [13].

**Theorem 1.6.** Let $q$ be a nonnegative integer and let $d = 2^t$ where $t$ is a positive integer. Then, there exists a $Q_d$-decomposition of $K_{2^{t+1}+qd,x2^{t-1}}$ for all positive integers $x$.

**Theorem 1.7.** If $d \geq 2$, then there exists a $Q_d$-decomposition of $K_{qd2^{t-1},yd2^{t-1}}$ for all positive integers $x$ and $y$.

**Corollary 1.8.** Let $d$, $m$ and $n$ be positive integers with $m \leq n$. If $n \equiv m \equiv 1 \pmod{d2^d}$, then there exists a $d$-cube decomposition of $K_m$ embedded in a $d$-cube decomposition of $K_n$. 