A New Recursive Theorem on $n$-Extendibility

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Abstract. A graph $G$ having a 1-factor is called $n$-extendible if every matching of size $n$ extends to a 1-factor. Let $G$ be a 2-connected graph of order $2p$. Let $r > 0$ and $n > 0$ be integers such that $p - r > n + 1$. It is shown that if $G \setminus S$ is $n$-extendible for every connected subgraph $S$ of order $2r$ for which $G \setminus S$ is connected, then $G$ is $n$-extendible.

We deal only with finite simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A \subset V(G)$, $G \setminus A$ is the subgraph of $G$ induced by $V(G) \setminus A$. If a subset $A$ induces a connected subgraph, then $A$ is said to be connected. Moreover, if $A$ is an empty set, then it is considered to be connected. If $A$ and $B$ are disjoint subsets of $V(G)$, then $E(A, B)$ denotes the set of edges with one end in $A$ and the other in $B$. If $H$ is a subgraph and $v$ is a vertex, we may write $E(v, H)$ and $G \setminus H$ instead of $E(\{v\}, V(H))$ and $G \setminus V(H)$, respectively. For $v \in V(G)$, $N_G(v)$ denotes the neighborhood of $v$. For $e \in E(G)$, $V'(e)$ is the set of endvertices of $e$. Furthermore, let $M$ be a matching (a set of independent edges) of $G$. $V'(M)$ denotes $\bigcup_{e \in M} V'(e)$. Let $n \geq 0$ and $p > 0$ be integers with $n \leq p - 1$ and let $G$ be a graph with $2p$ vertices having a 1-factor (a perfect matching). Then $G$ is said to be $n$-extendible if every matching of size $n$ in $G$ extends to a 1-factor. In particular, $G$ is 0-extendible iff $G$ has a 1-factor. Furthermore, $G$ is said to be $(r, n)$-extendible if every connected subset $S$ of order $2r$ is $n$-extendible. And $G$ is said to be $[r, n]$-extendible if $G \setminus S$ is $n$-extendible for every connected subset $S$ of order $2r$. Other terminology on graphs can be found in [1].

In [5], we proved the following recursive theorems on extendibility.

Theorem A. Let $r$ and $n$ be integers with $r > n > 0$. Then every $(r, n)$-extendible graph is $(r + 1, n + 1)$-extendible.

Theorem B. Let $p$, $r$ and $n$ be integers with $r > 0$ and $p - r > n \geq 0$. Then every connected $[r, n]$-extendible graph of order $2p$ is $[r - 1, n]$-extendible.

Note that a connected graph $G$ of even order is $(|G|/2, n)$-extendible iff $G$ is $n$-extendible. Furthermore, a graph $G$ is $[0, n]$-extendible iff $G$ is $n$-extendible. Therefore these theorems are natural extensions of the following theorems in [3] and [4], respectively.
Theorem C. Let $G$ be a connected graph of order $2p$ ($p \geq 3$), and let $r$ and $n$ be integers such that $1 \leq n < r < p$. If for some integer $r$, every induced connected subgraph of order $2r$ is $n$-extendible, then $G$ is $n$-extendible.

Theorem D. Let $G$ be a connected graph of order $2p$. Let $r$ and $n$ be positive integers such that $p - r \geq n + 1$. If $G \setminus S$ is $n$-extendible for every connected subset $S$ of order $2r$, then $G$ is $n$-extendible.

The motivation for Theorems A–D is the following theorem due to Sumner [7] on the existence of a 1-factor in a graph in terms of the existence of 1-factors in certain subgraphs.

Theorem E. Let $G$ be a connected graph of order $2p$ ($p > 1$). If for some integer $r$ with $1 < r \leq p$ every connected induced subgraph of $G$ of order $2r$ has a 1-factor, then $G$ has a 1-factor.

Very recently, Enomoto [2] weakened the condition of the above theorem for a 2-connected graph.

Theorem F. Let $p$ and $r$ be integers with $0 < r < p - 1$ and let $G$ be a 2-connected graph of order $2p$. If $G \setminus S$ has a 1-factor for every connected subset $S$ of order $2r$ for which $G \setminus S$ is connected, then $G$ has a 1-factor.

Moreover, Enomoto showed that all conditions of Theorem F are weakest possible. The purpose of this paper is to present a similar recursive result on $n$-extendibility that is an extension of Theorems C and D.

Theorem 1. Let $G$ be a 2-connected graph of order $2p$. Let $r \geq 0$ and $n > 0$ be integers such that $p - r \geq n + 1$. If $G \setminus S$ is $n$-extendible for every connected subset $S$ of order $2r$ for which $G \setminus S$ is connected, then $G$ is $n$-extendible.

Note that even when $n = 0$, the above theorem is not exactly the same as Theorem F. Theorem F requires $p \geq r + 2$, whereas $p \geq r + 1$ in Theorem 1 if $n = 0$. Moreover, the connectedness condition cannot be weakened. For example, let $K_{2n+2}$ and $K'_{2n+2}$ be two disjoint complete graphs with order $2n + 2$. Let $u \in V(K_{2n+2})$ and $v \in V(K'_{2n+2})$. Add an edge $uv$ between $K_{2n+2}$ and $K'_{2n+2}$. Let $G$ be the resulting graph. Now we can easily check that $G \setminus S$ is $n$-extendible for every connected subset $S$ of order $2n + 2$ for which $G \setminus S$ is connected. It is obvious however that $G$ is not $n$-extendible since $G$ cannot have a 1-factor which contains $uv$.

A graph $G$ is called $(r, n)$-extendible if $G \setminus S$ is $n$-extendible for every connected subset $S$ of order $2r$ for which $G \setminus S$ is connected. From this definition it is seen that if a connected graph is $[r, n]$-extendible or $(|G|/2 - r, n)$-extendible, then it is $(r, n)$-extendible and that a graph is $(0, n)$-extendible if it is $n$-extendible. Thus, to prove Theorem 1, it suffices to show the following theorem, which is an extension of Theorem B.

Theorem 2. Let $p$, $r$ and $n$ be integers with $r > 0$ and $p - r > n > 0$. Then every 2-connected $(r, n)$-extendible graph of order $2p$ is $(r - 1, n)$-extendible.