SPECTRAL REPRESENTATION OF A GENERALIZED TRACE AND DETERMINANT

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In the present paper we extend the notion of trace and determinant for wider sets of compact operators than the set of trace class operators. We express the trace and the determinant via the eigenvalues of the operator. As an application, we obtain a spectral expression for \( \det L(\lambda) \) where

\[
L(\lambda) = I - \sum_{j=1}^{n} \lambda^j H_j, \quad \text{with } H_j \in S_1, \quad j \geq 1.
\]

\[S1. \] INTRODUCTION

In this paper we introduce the notion of trace and determinant for classes of compact operators larger than the trace class. The main aim of this paper is to express this trace and determinant via the eigenvalues of the operator. We obtain formulas which generalize the formulas for trace class operators. As an application, an expression of \( \det(I-\lambda H_1 \ldots - \lambda^n H_n) \) with trace class coefficients \( H_j, \quad j \geq 1, \) is given in terms of the characteristic numbers of the polynomial \( L(\lambda) = I - \sum_{j=1}^{n} \lambda^j H_j. \) This result is obtained in section four.

To make the construction clearer, we first discuss the case \( n=2. \) This is done in section 2.

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Throughout the paper, we denote by \( S_p (0 \leq p \leq \infty) \), the ideal of all compact operators \( A, \) acting in a separable Hilbert space \( H, \) with sequence of singular numbers \( s_1 \geq s_2 \geq \ldots \) satisfies \( \sum_{j=1}^{\infty} s_j^p < \infty, \) with the usual convention for \( p=\infty. \) The \( p \)-norm of \( A \in S_p \) is
Denote by $\sigma(A)$ the spectrum of $A$.

§2. THE CASE OF BLOCK MATRICES OF SIZE TWO

Denote by $T_2$ the set of all the block matrices $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ acting on $H = H_1 \oplus H_2$. Here $H_1$ and $H_2$ are Hilbert spaces, and the norm on $H$ is $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$, $x_1 \in H_1$ and $x_2 \in H_2$. We also assume that $A_1, A_3 \in S_1$ and $A_2, A_4 \in S_2$. Further we introduce a norm on $T_2$ by the following rule

$$|A|_{T_2} = \|A_1\|_1 + \|A_2\|_2 + \|A_3\|_1 + \|A_4\|_2, \quad A \in T_2.$$  

Let $K_n, n \geq 1,$ be a sequence of finite rank operators, which tends to $A$ in the $T_2$-norm. We call $K_n, n \geq 1,$ a representative sequence for $A$. Define $\text{tr}A = \text{tr}A_1 + \text{tr}A_3$ and $\det(I-\lambda A) = \lim_{(n \to \infty)} \det(I-\lambda K_n), A \in T_2$. We shall see that the determinant $\det(I-\lambda A)$ is an entire function, and if we denote by $1 - \sum_{i=1}^{\infty} \lambda^i a_i$ the series expansion of $\det(I-\lambda A)$, then $\text{tr}A = -a_1$. But first let us prove that $\det(I-\lambda A)$ is well defined.

**Lemma 2.1:** Let $A \in T_2$ and let $K_n, n \geq 1,$ be a representative sequence for $A$. Then the sequence $\det(I-\lambda K_n)$ is uniformly convergent on every compact subset of $\mathbb{C}$, and the limit is an entire function which depends only on the operator $A$ and not on the representative sequence $K_n$.

**Proof:** Since $\det(I-\lambda K_n)$ is a sequence of entire functions, it is sufficient to show that it is a Cauchy sequence uniformly on every compact set. If $\lambda \notin \sigma(A_3)$ we can factor the operator $I-\lambda A$ in the following way: