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Generic Instability of Rotating Relativistic Stars

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Abstract. All rotating perfect fluid configurations having two-parameter equations of state are shown to be dynamically unstable to nonaxisymmetric perturbations in the framework of general relativity. Perturbations of an equilibrium fluid are described by means of a Lagrangian displacement, and an action for the linearized field equations is obtained, in terms of which the symplectic product and canonical energy of the system can be expressed. Previous criteria governing stability were based on the sign of the canonical energy, but this functional fails to be invariant under the gauge freedom associated with a class of trivial Lagrangian displacements, whose existence was first pointed out by Schutz and Sorkin [12]. In order to regain a stability criterion, one must eliminate the trivials, and this is accomplished by restricting consideration to a class of "canonical" displacements, orthogonal to the trivials with respect to the symplectic product. There nevertheless remain perturbations having angular dependence $e^{im\phi}$ (ϕ the azimuthal angle) which, for sufficiently large m, make the canonical energy negative; consequently, even slowly rotating stars are unstable to short wavelength perturbations. To show strict instability, it is necessary to assume that time-dependent nonaxisymmetric perturbations radiate energy to null infinity. As a byproduct of the work, the relativistic generalization of Ertel's theorem (conservation of vorticity in constant entropy surfaces) is obtained and shown to be Noether-related to the symmetry associated with the trivial displacements.

I. Introduction

In the introduction to their 1970 paper on cosmological singularities, Hawking and Penrose [1] noted that gravity is an "essentially unstable" force. For small concentrations of mass, the instability is masked by enormously larger short range forces. But when the density of matter is sufficiently large or its mass sufficiently great, gravity becomes dominant and collapse inevitable. From this instability to collapse arises the theoretical expectation of black holes; and the strongest observational argument in their favor is provided by the associated upper limit on

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the mass of dense spherical stars. Rotation can delay the onset of instability to radial pulsation and in this way raise the upper mass limit. Remarkably, however, gravity seems to provide a conspiracy of instabilities: we will find that any rotating, self-gravitating perfect fluid, is unstable to nonaxisymmetric perturbations, which (presumably) radiate its angular momentum until it settles down to a non-rotating star.

Dynamical instability of rotating Newtonian stars was first understood for the Maclaurin sequence of uniformly rotating, uniform density ellipsoids (see [2]). For sufficiently rapid rotation, the sequence becomes unstable to nonaxisymmetric deformations, via a mode having angular dependence (in the linear theory) $e^{2i\phi}$, where ϕ is the angle about the symmetry axis. When viscosity is present, instability sets in earlier – for smaller angular momentum, but again via an $m=2$ mode [3, 4]. A parallel situation in relativity was first considered by Chandrasekhar [5]. Using a post-Newtonian treatment, he found that in the presence of radiation reaction, an $m=2$ mode again becomes unstable at precisely the same point along the sequence that marks the onset of viscosity-induced instability. It has subsequently been widely assumed that the Maclaurin sequence, and rotating fluids in general, are stable in relativity for small values of the angular momentum as is the case in the strictly Newtonian theory. Recently, however, Schutz and I [6] showed, in a post-Newtonian framework, that the radiation reaction induced instability sets in first via short wavelength oscillations and that even for slowly rotating configurations, there are, for sufficiently large m, unstable modes having angular dependence $e^{\mathrm{i}m\phi}$; and Comins [7] has explicitly found the corresponding unstable modes of the slowly rotating Maclaurin ellipsoids. The present paper extends these post-Newtonian results to the exact theory, showing that all rotating axisymmetric perfect fluid configurations are unstable to perturbations having angular dependence of the form $e^{\mathrm{i}m\phi}$ for all integers m greater than some m_0 [1]. Strictly, we establish the existence of at best marginally stable perturbations. A proof of instability requires the additional assumption that time dependent, nonaxisymmetric oscillations of an axisymmetric star radiate, at least when m is large – presumably when $m>1$; although the assumption may appear obvious, in that the multipole moments must change at null infinity, there is as yet no formal proof.

A side issue in the work involves the clarification of an oversight that plagued nearly all recent studies involving the stability of rotating stars (see [6] and [8]), and which arises in the following way. In phrasing a stability criterion, one introduces a canonical energy [9–11], obtained from the action that governs the linear perturbation equations [2]. The existence of an action is predicated on a description of fluid perturbations in terms of a Lagrangian displacement, a vector field connecting fluid elements in the perturbed and unperturbed flows (equivalently, one must single out a "comoving frame"). But such a description is not unique: Schutz and Sorkin [12] have pointed out the existence of a class of trivial displacements that leave invariant all physical quantities, so that a given physical

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1 The work here is restricted to fluids having two parameter equations of state. The generic instability for strictly isentropic fluids differs somewhat in its physical features and its mathematical details

2 In references [9] and [10], the stability functionals are equivalent to the canonical energy of time-independent initial data, although they are not so identified