TOEPLITZ AND HANKEL MATRICES ON $\mathbb{C}^n$

TAKASHI YOSHINO

Dedicated to Professor Masanori Fukamiya on his 88th birthday

It is known that the characterizations of the Toeplitz operator $T_\varphi$ on $H^2$ and also the Hankel operator $H_\varphi$ on $H^2$ by using the simple unilateral shift $T_2$. Recently, some characterization of the normal Toeplitz matrix truncated on $\mathbb{C}^n$ is given by D.R. Farenick, M. Krupnik, N. Krupnik and W. Y. Lee [1] and, independently, by T. Ito [2]. In this paper we shall give some characterizations of the Toeplitz matrix and also the Hankel matrix truncated on $\mathbb{C}^n$.

Let

$$V_n = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
1 & \ddots & & & \\
0 & \ddots & & & \\
\vdots & & \ddots & & \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}$$

Then the Toeplitz matrices on $\mathbb{C}^n$ are characterized as follows.

Theorem 1. The following assertions are equivalent.

1. $A_n \in B(\mathbb{C}^n)$ is a Toeplitz matrix.
2. $V_n V_n^* A_n V_n V_n^* = V_n A_n V_n^*$.
3. $V_n V_n^* A_n V_n V_n^* V_n = V_n^* A_n V_n^* V_n$.

Proof. By the simple calculation of matrices, we see that $V_n V_n^* A_n V_n V_n^* = V_n A_n V_n^*$ if and only if $A_n$ is a Toeplitz matrix. If $V_n V_n^* A_n V_n V_n^* = V_n A_n V_n^*$, then

$$V_n^* (V_n V_n^* A_n V_n V_n^*) V_n = V_n^* (V_n A_n V_n^*) V_n.$$

Since $V_n$ is a partial isometry, $V_n^* V_n V_n^* = V_n^*$ and $V_n V_n^* V_n = V_n$ and hence $V_n^* A_n V_n = V_n^* V_n A_n V_n^* V_n$. Conversely, if $V_n^* A_n V_n = V_n^* V_n A_n V_n^* V_n$, then, by the same reasons as above, $V_n (V_n^* A_n V_n) V_n^* = V_n (V_n^* V_n A_n V_n V_n^*) V_n^* = V_n A_n V_n^*$. □
Lemma 1. For any fixed $\theta \in [0, 2\pi)$, $V_n + e^{i\theta}V_n^{*-1}$ is an unitary Toeplitz matrix.

Proof. It is clear that $V_n + e^{i\theta}V_n^{*-1}$ is a Toeplitz matrix. Since $V_n^* = O$,

\[
(V_n + e^{i\theta}V_n^{*-1})^* (V_n + e^{i\theta}V_n^{*-1}) = (V_n^* + e^{-i\theta}V_n^{*-1})(V_n + e^{i\theta}V_n^{*-1}) = V_n^* V_n + V_n^{*-1} V_n^{*-1} = I_n
\]

Therefore we have the conclusion by the mathematical induction. \(\square\)

Lemma 2. For any fixed $\theta \in [0, 2\pi)$, $\left(V_n + e^{i\theta}V_n^{*-1}\right)^j = V_n^j + e^{i\theta}V_n^{*-j}$ \quad (j = 1, 2, \cdots, n).

Proof. It is clear that the equation holds in the case where $j = 1$. And, for a fixed $j$ such as $1 \leq j < n$, let $(V_n + e^{i\theta}V_n^{*-1})^j = V_n^j + e^{i\theta}V_n^{*-j}$. Then

\[
(V_n + e^{i\theta}V_n^{*-1})^{j+1} = (V_n + e^{i\theta}V_n^{*-1})(V_n^j + e^{i\theta}V_n^{*-j}) = V_n^{j+1} + e^{i\theta} \left\{ V_n^{j-1} V_n^j + V_n^{*-1} V_n^{*-j} \right\} \quad \text{(because $V_n^* = O$)}
\]

Therefore we have the conclusion by the mathematical induction. \(\square\)

Theorem 2. For $A_n \in \mathcal{B}(C^n)$ and for any fixed $\theta \in [0, 2\pi)$, if $A_n$ commutes with $V_n + e^{i\theta}V_n^{*-1}$, then $A_n$ is a Toeplitz matrix.

Proof. If $(V_n + e^{i\theta}V_n^{*-1})A_n = A_n (V_n + e^{i\theta}V_n^{*-1})$, then

\[
V_n^* \left[(V_n + e^{i\theta}V_n^{*-1})A_n\right] V_n^* = V_n^* \left[A_n (V_n + e^{i\theta}V_n^{*-1})\right] V_n
\]

and $V_n^*V_n A_n V_n^* V_n = V_n^* A_n V_n^* V_n = V_n^* A_n V_n$ because $V_n^{*-1} = O$ and hence, by Theorem 1, $A_n$ is a Toeplitz matrix. \(\square\)

Since

\[
V_n + e^{i\theta}V_n^{*-1} = \begin{pmatrix}
0 & \cdots & 0 & e^{i\theta} \\
1 & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]

n times