PERTURBED PROJECTION METHODS FOR SPLIT EQUATIONS OF THE FIRST KIND

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The paper investigates the approximate solution of linear operator equations of type \((J-K)u = v\), with \(J\) assumed invertible, obtained by applying a linear projection on both sides of the equation, together with a linear perturbation operator on the left side and a perturbation element on the right side.

1. INTRODUCTION. Given Banach spaces \(E\) and \(F\), we denote by \(L(E,F)\) the Banach space of bounded linear operators \(L : E \rightarrow F\). As usual, we denote \(L(F,F)\) by \(L(F)\). Consider an operator equation

\[(J-K)u = v,\]

where \(J, K \in L(E,F)\) and \(v \in F\) are given. We assume throughout that \(J^{-1}\) exists in \(L(F,E)\). Consider

\[E_n\] a closed subspace of \(E\),

\[F_n = \mathfrak{J}E_n\] a corresponding closed subspace of \(F\),

\[P_n : F \rightarrow F_n\] a linear projection of \(F\) onto \(F_n\) not assumed bounded, however \(P_n K\) is assumed bounded,

\[Q_n = J^{-1}P_n J\] a corresponding linear projection of \(E\) onto \(E_n\),

\[R_n \in L(E_n,F_n)\] a perturbation operator,

\[r_n \in F_n\] a perturbation element.
We deal with the approximate equation
\[(J-P_nK-R_n)u_n = P_n v + r_n, \quad (u_n \in E_n). \tag{2}\]
The inclusion in (2) means that a solution \(u_n\) is sought in the subspace \(E_n\).

If \(E = F\) and \(J = I\) then (1) becomes an equation of the second kind,
\[x = Mx + v, \tag{3}\]
with \(M \in L(F)\) and \(v \in F\) given. The corresponding approximate equation becomes
\[x_n = (P_nM + S_n)x_n + P_n v + r_n, \quad (x_n \in F_n), \tag{4}\]
where \(P_n, r_n\) are as before and \(S_n \in L(F_n)\) denotes a perturbation operator.

Vainikko [3] studied nonlinear analogues of (3) and (4) in order to analyze quadrature methods for nonlinear Fredholm integral equations. The corresponding results for the linear case are explicitly stated in Krasnosel'skiii et al [2, Section 17]. If \(R_n = 0\) and \(r_n = 0\) then (2) represents a projection method readily related to a projection method for an equation of the second kind by taking \(x_n = Ju_n, M = KJ^{-1}\), and \(S_n = 0\) in (4); see [2, Section 15.6]. Our goal is to analyze (2) with nonzero perturbations by relating it to (4) with corresponding nonzero perturbations. Occurrences of (2) in the numerical analysis of important problems warrant an explicit formulation of the results.

2. A CONVERGENCE RESULT. In the sequel, we let \(M_n = P_nM - S_n, P(n) = I - P_n, Q(n) = I - Q_n\), and for the sake of clarity, we occasionally indicate the spaces, under the symbol \(\| \cdot \|\), over which norms are taken. The following lemma, based on two applications of the Banach Lemma, gives sufficient conditions under which the well-posedness of equation (3) implies that of the approximate equation (4).

**Lemma 1.** If \(I - M\) has an inverse in \(L(F)\) with \(\| (I-M)^{-1} \|_{L(F)} \leq \alpha\) and \(\alpha(\| M - P_n M \|_{L(F)} + \| S_n \|_{L(F_n)}) \leq \beta_n < 1\) then \(I - M_n\) has an inverse in \(L(F_n)\)