EXPLICIT WIENER-HOPF FACTORIZATION
FOR CERTAIN NON-RATIONAL MATRIX FUNCTIONS

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Explicit Wiener-Hopf factorizations are obtained for a certain class of nonrational
2 x 2 matrix functions that are related to the scattering matrices for the 1-D Schrödinger
equation. The diagonal elements coincide and are meromorphic and nonzero in the upper-
half complex plane and either they vanish linearly at the origin or they do not vanish.
The most conspicuous nonrationality consists of imaginary exponential factors in the off-
diagonal elements.

1. INTRODUCTION

In this article we obtain explicit Wiener-Hopf factorizations of certain nonrational
2 x 2 matrix functions which arise as (modified) scattering matrices for the 1-D Schrödinger
equation [20,21,22] and some related Schrödinger-type equations [6,8]. These matrix
functions have the form

\[ G(k, x) = \begin{pmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{pmatrix} \]

where, for any real parameter \( x \),

1. \( T(k) \) is nonzero on \( \mathbb{C}^+ \setminus \{0\} \),
2. \( R(k) \) and \( L(k) \) are meromorphic on \( \mathbb{C}^+ \) with continuous boundary values
   on the extended real axis and vanish as \( k \to \infty \) in \( \mathbb{C}^+ \),
3. \( G(k, x)^{-1} = qG(-k, x)q \) for \( k \in \mathbb{R} \), where \( q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and
4. \( G(k, x) \), as a function of \( k \in \mathbb{R} \), belongs to a suitable Banach algebra of 2 x 2 matrix
   functions within which Wiener-Hopf factorization is possible. This may be the Wiener
   algebra or the algebra of functions \( f(k) \) such that \( f^*(\xi) = f(i^{1+\xi}) \) is Hölder continuous
   with exponent \( \alpha \) on the unit circle where \( \alpha \in (0, 1) \). We will define these algebras
   shortly.

Wiener-Hopf factorization problems of the above type arise as an offshoot of the
inverse scattering problems for the 1-D Schrödinger equation [20,21,22] and some related

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Footnote:
1 Throughout this article we denote by \( \mathbb{C}^+ \) and \( \mathbb{C}^- \) the open upper and lower half-planes
and by \( \overline{\mathbb{C}^+} \) and \( \overline{\mathbb{C}^-} \) the closed upper and lower half-planes including infinity.
Schrödinger-type equations [6,8]. $T(k)$ is usually called the transmission coefficient, $R(k)$ and $L(k)$ the reflection coefficients from the right and the left, respectively, and

\begin{equation}
S(k) = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix}
\end{equation}

is the scattering matrix. The solution of the inverse scattering problem is achieved by obtaining the potential of the Schrödinger equation when the scattering matrix is known. Such an inverse scattering problem can be posed [20,21,22] as a Riemann-Hilbert problem which can be solved by various means, such as the methods due to Gel'fand and Levitan, Marchenko, Faddeev, and Newton [11,12,13,20,21,22], where the Riemann-Hilbert problem is transformed into a nonhomogeneous Fredholm integral equation. When the reflection coefficients have meromorphic extensions to $\mathbb{C}^+$, the resulting integral equation has a separable kernel and thus its solution can be obtained explicitly by solving a system of linear algebraic equations. It is then possible to obtain the solution of this Riemann-Hilbert problem by a contour integration [1] without solving the Fredholm integral equation when $T(0) \neq 0$; if $T(0) = 0$, one can find a scattering matrix $S_\epsilon(k)$ such that its transmission coefficient does not vanish at $k = 0$ and $S_\epsilon(k) \rightarrow S(k)$ as $\epsilon \rightarrow 0$. Then the Riemann-Hilbert problem can be solved using $S_\epsilon(k)$ as the input matrix, and then letting $\epsilon \rightarrow 0$ one obtains the solution of the Riemann-Hilbert problem where the input matrix is $S(k)$ [1,2]. When $T(k)$ has a zero at $k = 0$, the factorization of $G(k, x)$ becomes noncanonical; in this case the solution of the inverse scattering problem becomes nonunique unless $R(0) = L(0) = -1$ and the zero of $T(k)$ at $k = 0$ is a simple one. Explicit examples of nonuniqueness of the solution of the inverse scattering problem for the 1-D Schrödinger equation can be found in [3,4,5,7,10].

For many years it has been customary to view explicit Wiener-Hopf factorization of nonrational matrix functions as a Herculean task well-nigh impossible to carry out. In recent years there have appeared some papers [16,17,19,24] in which nonrational $2 \times 2$ matrix functions within special classes are factorized explicitly. The present article is devoted to a completely different class of $2 \times 2$ matrix functions and our factorization method differs significantly from the ones adopted in [16,17,19,24]. In this paper we will obtain the Wiener-Hopf factors of the matrix $G(k, x)$ given in (1.1) by the contour integration method.

This article is organized as follows. In Section 2 we give the preliminary results needed for the factorization. In Section 3, assuming $T(0) \neq 0$, we pose the inverse scattering problem for the 1-D Schrödinger equation as a matrix Riemann-Hilbert problem and obtain the canonical Wiener-Hopf factors of $G(k, x)$ by solving the Riemann-Hilbert problem posed. In Section 4 explicit canonical factorizations of $G(k, x)$ are obtained by the contour integration method when the reflection coefficients have meromorphic extension to $\mathbb{C}^+$ with continuous boundary values as $k$ approaches the extended real axis. In Section 5 we treat the case $T(0) = 0$ and the case where the extension of $T(k)$ to $\mathbb{C}^+$ is meromorphic, and we obtain the noncanonical Wiener-Hopf factorization of $G(k, x)$. In Section 6 some instructive examples are presented. Finally, in the Appendix some special functions needed in Section 4 are defined.

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