A GENERALIZATION OF ARCANGELI'S METHOD FOR
ILL-POSED PROBLEMS LEADING TO OPTIMAL RATES

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Schock (1984) considered a general a posteriori parameter choice strategy for the regularization of ill-posed problems which provide nearly the optimal rate of convergence. We improve the result of Schock and give a class of parameter choice strategies leading to optimal rates. As a particular case we prove that the Arcangeli's method do give optimal rate of convergence.

Let $X$ and $Y$ be Hilbert spaces and $T : X \to Y$ be a bounded linear operator with non-closed range $R(T)$. We consider the regularization of the ill-posed operator equation

\[ (1) \quad Tx = y, \quad y \in R(T), \]

with an inexact data $y^\delta$ in place of $y$, where $\|y - y^\delta\| \leq \delta, \delta > 0$. Let $\hat{x}$ be the unique element in $N(T)^\perp$, the orthogonal compliment of the null space $N(T)$ of $T$, satisfying the equation (1). For $\alpha > 0$, let $x$ be the minimizer of the functional

\[ \min_{u \in X} \|Tu - y^\delta\|^2 + \alpha \|u\|^2, \quad u \in X. \]

It is known that (Schock [4])

\[ x_\alpha^\delta = (T^*T + \alpha I)^{-1}T^*y^\delta, \]

and if $\hat{x} \in R((T^*T)^r)$, $0 < r \leq 1$, then

\[ (2) \quad \|x - x_\alpha^\delta\| \leq c_1 \alpha^r + c_2 \delta \sqrt{\alpha}. \]
Here, and below, $c_1$, $c_2$, etc. are positive generic constants which may take different values at different contexts.

An important point under consideration is to choose $a = a(\delta)$ such that

$$| | \hat{x} - x^\delta_a | | \to 0 \text{ as } \delta \to 0,$$

and to obtain $\alpha^r$ and $\delta / \sqrt{\alpha}$ in terms of the powers of $\delta$. Arcangeli [1] suggested the 'discrepancy principle'

$$| | T x^\delta_a - y^\delta | | = \delta / \sqrt{\alpha}$$

for the choice of $\alpha = \alpha(\delta)$, and showed that the equation (4) has a unique solution and that (3) is satisfied. It is proved by Groetsch and Schock [3] that if $x \in R(T^*)$ and $\alpha$ satisfies (4) then $| | \hat{x} - x^\delta_a | | = O(\delta^{1/3})$. It is also proved in [3] that under (4) the best possible rate is $O(\delta^{2/3})$. It is not known so far that the rate $O(\delta^{2/3})$ can be attained except when $R(T)$ is closed (c.f. [3]). It is one of the aims of this paper to settle this problem positively by showing that if $\hat{x} \in R(T^*T)$ and $\alpha = \alpha(\delta)$ is chosen according to (4), then we have the rate $O(\delta^{2/3})$.

We consider the general discrepancy principle

$$| | T x^\delta_a - y^\delta | | = \delta^p / a^q$$

suggested by Schock [5] for $p > 0$, $q > 0$. Choosing $\alpha = \alpha(\delta)$ according to (5) Schock [5] proved the following result.

**THEOREM 1** (Schock [5]). Let $\hat{x} \in R((T^*T)^r)$, $0 < r \leq 1$, let for $q > 0$, $p = 4q(q+1)/(4rq+2q+1)$, $k = 4rq/(4rq+2q+1)$ and let $\alpha = \alpha(\delta)$ be chosen according to (5). Then

$$| | \hat{x} - x^\delta_a | | = O(\delta^k).$$

We improve this result of Schock. First we state without proof the following result in [5].