An interior-point QP algorithm for structural optimization

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Abstract In this paper, a new algorithm for convex quadratic programming (QP) is presented. Firstly, the surrogate problem for QP is developed, and the Karush-Kuhn-Tucker conditions of the surrogate problem hold if the unconstrained minimum of the objective function does not satisfy any constraints. Then, Karmarkar's algorithm for linear programming (LP) is introduced to solve the surrogate dual problem. In addition, the case of general constraints is also discussed, and some examples of optimum truss sizing problems show that the proposed algorithm is robust and efficient.

1 Introduction

The work of Patnaik et al. (1996) shows that the sequential quadratic programming (SQP) algorithm is reliable and robust for large scale structural optimization. A series of QP subproblems should be solved in the SQP algorithm, so it is important to improve the algorithm for QP. Karmarkar (1984) introduced an interior-point method for linear programming. The algorithm generates a sequence of points in the interior of the feasible region which converges to the optimal solution. The most important and useful part of Karmarkar's algorithm is the projective scaling transformation. It makes a sequence of points in the interior of the feasible region. In addition, one constraint in Karmarkar's canonical LP form is that the sum of all variables is one, i.e. \( \sum_{i=1}^{n} x_i = 1 \), which is termed the simplex. This makes us think of the surrogate multipliers of surrogate problems (Glover 1968; Greenberg and Pierskalla 1970), which are of the same form as that of the variables of Karmarkar's canonical form. The surrogate approach can be used to solve inequality constrained optimization problems. Although the surrogate theory was established more than twenty years ago, few efficient algorithms have been developed, because it is difficult to derive an explicit surrogate dual problem. For the QP problem, the explicit surrogate dual problem can be derived under certain conditions, and then Karmarkar's algorithm can be used to solve the surrogate dual problem.

2 Surrogate approach to constrained optimization

A surrogate problem of mathematical programming is one in which the original constraints are replaced by only one constraint, termed the surrogate constraint, which is a positive linear combination of the original constraints. The concept of a surrogate constraint was first introduced by Glover (1968), who employed it in solving integer programming problems. Greenberg and Pierskalla (1970) provided the first major theoretical treatment of surrogate constraints in the context of general mathematical programming.

For the following inequality constrained problem, referred to as the primal problem,

\[
\begin{align*}
\text{minimize} & \quad f(x), \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, \ldots, m, \\
& \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( x \in \mathbb{R}^n \), the corresponding surrogate problem has the form

\[
\begin{align*}
\text{minimize} & \quad f(x), \\
\text{subject to} & \quad \sum_{j=1}^{m} \lambda_j g_j(x) \leq 0, \\
& \quad \lambda_j (j = 1, \ldots, m) \text{ are nonnegative weights termed surrogate multipliers which may be normalized without loss of generality by requiring} \\
& \quad \sum_{j=1}^{m} \lambda_j = 1.
\end{align*}
\]

Some relationship between problems (P) and (S) is presented below as a proposition.

Proposition 2.1.

1. The feasible region of problem (P) is always included by the feasible region of problem (S).
2. If \( x^s \) solves problem (S) and \( x^* \) solves problem (P), then \( f(x^s) \leq f(x^*) \) for all \( \lambda_j \geq 0, \ j = 1, \ldots, m \).
3. If \( x^s \) solves problem (S) and is also feasible in problem (P), then \( x^s \) also solves problem (P).

This proposition can easily be verified.

A surrogate dual objective can be defined as

\[
s(\lambda) = \left\{ \min f(x) : \sum_{j=1}^{m} \lambda_j g_j(x) \leq 0 \right\},
\]

which is the minimum value of \( f(x) \) at the solution point of problem (S) for fixed \( \lambda > 0 \), which of course satisfies the normality condition (1). Then the surrogate dual problem is obtained,
max \, s(\lambda),
\begin{align*}
\text{subject to } \\
\sum_{j=1}^{m} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \ldots, m.
\end{align*} \quad (SD)

Problem (SD) can be written in the following form:

\begin{align*}
\max \min_{\lambda} \left\{ f(x) + \alpha \sum_{j=1}^{m} \lambda_j g_j(x) \right\}, \\
\text{subject to } \\
\sum_{j=1}^{m} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \ldots, m,
\end{align*}

where \( \alpha \) is the Lagrange multiplier associated with the surrogate constraint \( \sum_{j=1}^{m} \lambda_j g_j(x) \leq 0 \).

One advantage is that there is no gap between problems (P) and (SD) for the convex objective function (Greenberg and Pierskalla 1970).

\section{3 Surrogate problem of quadratic programs}

In this paper, we consider the following quadratic problem:

\begin{align*}
\min \, f(x) = \frac{1}{2} x^T H x + c^T x, \quad (2a) \\
\text{subject to } A x - b < 0, \quad (2b)
\end{align*}

where \( x, c \in \mathbb{R}^n, H \in \mathbb{R}^{n \times n} \) is symmetric positive definite, \( A \in \mathbb{R}^{m \times n} \), rank \((A) = m, b \in \mathbb{R}^m, \) and \( 0 \) is the vector in which all components are zero. If the optimal solution of (2) is not inside the feasible region, it must be on the boundary of the feasible region. In this condition the active constraint can be expressed as a convex combination of all constraints, by means of the idea of surrogate constraints, (2) can be changed to

\begin{align*}
\min \, f(x) = \frac{1}{2} x^T H x + c^T x, \quad (3a) \\
\text{subject to } \\
\lambda^T (A x - b) \leq 0, \quad \lambda \in \Delta, \quad (3b)
\end{align*}

in which \( \lambda = (\lambda_1, \ldots, \lambda_m)^T \) is a surrogate multiplier vector, \( \Delta = \{ \lambda \in \mathbb{R}^m \mid e^T \lambda = 1, \lambda \geq 0 \}, e = (1, 1, \ldots, 1)^T \in \mathbb{R}^m \) and \( \Delta \) is referred to as simplex. The Lagrangian function is

\begin{equation}
L(x, \lambda, \alpha) = \frac{1}{2} x^T H x + c^T x + \alpha \lambda^T (A x - b), \quad (4)
\end{equation}

where \( \alpha \) is a Lagrange multiplier. The Karush-Kuhn-Tucker conditions of (3) are

\begin{align*}
H x + c + \alpha A^T \lambda &= 0, \quad (5a) \\
\lambda^T (A x - b) &\leq 0, \quad (5b) \\
\alpha &\geq 0, \quad (5c) \\
\alpha \lambda^T (A x - b) &= 0. \quad (5d)
\end{align*}

There are two cases obtained from the positive condition (5c) and the complementary condition (5d).

1. If \( \alpha = 0 \), then from (4) the unconstrained minimum \( x^0 = -H^{-1}c \) is the optimal solution of problem (3).
2. If \( \alpha \neq 0 \), then \( \lambda^T (A x - b) = 0. \)

We mainly discuss the second case, where (5b) becomes an equal equation,

\begin{equation}
\lambda^T (A x - b) = 0. \quad (6)
\end{equation}

With (5a) and (6) the following results can be obtained:

\begin{align*}
\alpha &= -\frac{\lambda^T (A H^{-1} c + b)}{\lambda^T A H^{-1} A^T \lambda}, \quad (7) \\
x &= -H^{-1} (c + \alpha A^T \lambda). \quad (8)
\end{align*}

Note that \( \alpha > 0 \) cannot be guaranteed even if the complementary condition (6) holds. Because rank \((A) = m, A H^{-1} A^T \) is positive definite, and \( \lambda^T A H^{-1} A^T \lambda > 0 \), and if \( x^0 = -H^{-1} c \) violates any constraint of (2), \( A x^0 - b > 0 \), that is \( A H^{-1} c + b < 0 \), so \( \lambda^T (A H^{-1} c + b) < 0 \). From all of the above we have: if \( x^0 = -H^{-1} c \) violates any constraint of (2), then \( \alpha > 0 \).

The Karush-Kuch-Tucker conditions hold on the condition \( A x^0 > b \). Let \( d = A H^{-1} c + b, B = A H^{-1} A^T, \) and substitute (7) and (8) into (4), then we obtain the following surrogate dual problem. Let \( s(\lambda) \) denote the objective function

\begin{align*}
\max \, s(\lambda) &= \frac{1}{2} \lambda^T B \lambda - \frac{1}{2} c^T H^{-1} c, \quad (9a) \\
\text{subject to } \\
\lambda &\in \Delta. \quad (9b)
\end{align*}

This is the explicit surrogate dual problem. From the theory of Greenberg and Pierskalla (1970), there is no gap between (2) and (9), so we have the following theorem.

\textbf{Theorem 1.} Let \( x^* \) be the optimal solution of (2), if the unconstrained minimum \( x^0 \) does not satisfy any constraint of (2), and if \( \lambda^* \) is the optimal solution of (9), then \( f(x^*) = s(\lambda^*). \)

Let us use a simple example to show the geometrical explanation of the above conclusions,

\begin{align*}
\min \, x_1^2 + x_2^2 - 2x_1 - 5x_2, \\
\text{subject to } \\
-x_1 + x_2 \leq 1, \quad 3x_1 + x_2 \leq 3. \quad (P)
\end{align*}

The problem is shown in Fig. 1, \( P_i = (0.5, 1.5) \) is the optimal solution, \( c_1, c_2 \) is used to denote the two constraints, and \( A \) is the unconstrained minimum.

The surrogate constraint problem is

\begin{align*}
\min \, x_1^2 + x_2^2 - 2x_1 - 5x_2, \\
\text{subject to } \\
\lambda(-x_1 + x_2 - 1) + (1 - \lambda)(3x_1 + x_2 - 3), \quad 0 \leq \lambda \leq 1. \quad (S)
\end{align*}

It is simply found that, for a fixed \( \lambda \), the surrogate constraint is a hemiplane whose boundary is a line through \( P_i \), and the