Local Analysis of Newton-Type Methods for Variational Inequalities and Nonlinear Programming

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Abstract. This paper presents some new results in the theory of Newton-type methods for variational inequalities, and their application to nonlinear programming. A condition of semistability is shown to ensure the quadratic convergence of Newton's method and the superlinear convergence of some quasi-Newton algorithms, provided the sequence defined by the algorithm exists and converges. A partial extension of these results to nonsmooth functions is given. The second part of the paper considers some particular variational inequalities with unknowns \((x, \lambda)\), generalizing optimality systems. Here only the question of superlinear convergence of \(\{x^k\}\) is considered. Some necessary or sufficient conditions are given. Applied to some quasi-Newton algorithms they allow us to obtain the superlinear convergence of \(\{x^k\}\). Application of the previous results to nonlinear programming allows us to strengthen the known results, the main point being a characterization of the superlinear convergence of \(\{x^k\}\) assuming a weak second-order condition without strict complementarity.

Key Words. Variational inequalities, Nonlinear programming, Successive quadratic programming, Superlinear convergence.

AMS Classification. Primary 90C30, Secondary 65K05, 90C26.

1. Introduction

This paper is devoted to the local study of Newton-type algorithms for variational inequalities. Variational inequalities have been studied for a long time (see [16]) mainly because of their applications to mechanical systems. The operators in that
field are often monotone, and a large theory of monotone operators has been developed (see [6]); several algorithms for convex programming, including duality methods, have been extended to this framework (see [11]). Some problems in economy as well as optimality systems of nonlinear programming problems can also be represented by variational inequalities (see [21] and [13]). Consequently, the strength and large use of Newton-type algorithms for nonlinear programming, the so-called successive quadratic programming (see [2] and [10]), suggests developing a theory of Newton-type methods for variational inequalities (we do not speak here of some different approaches of Newton-type algorithms for variational inequalities—reviewed in the survey by Harker and Pang [13]). Some early (but unpublished) works in this direction due to Josephy [14], [15] give a local analysis using the concept of strong regularity [19]. Josephy obtains a quadratic rate of convergence for Newton’s method and superlinear convergence for some quasi-Newton algorithms. In the case of nonlinear programming problems, assuming the gradients of active constraints to be linearly independent, the strong regularity reduces to some strong second-order sufficient condition.

The quadratic rate of convergence under the weak second-order sufficiency condition for nonlinear programming problems, and assuming the linear independence of the gradients of active constraints, has been recently obtained by the author [4]. This suggests that the theory of Newton-type methods for variational inequalities can be extended. For this purpose we use the new concept of semistability. We say that a solution $\bar{x}$ of a variational inequality is semistable if, given a small perturbation on the right-hand side, a solution $x$ of the perturbed variational inequality that is sufficiently close to $\bar{x}$, is such that the distance of $x$ to $\bar{x}$ is of the order of the magnitude of the perturbation. This does not imply the existence of a solution for the perturbed problem. Indeed, we give a counterexample showing that existence for a small perturbation does not always hold under the semistability hypothesis. We use a “hemistability” hypothesis in order to prove the existence of the sequence satisfying the Newton-type steps, then we show that semistability allows us to obtain in a simple way quadratic convergence for Newton’s method and superlinear convergence for a large class of Newton-type algorithms (here we extend the Dennis and Moré [9] sufficient condition for superlinear convergence). This allows us to adapt Grzegórski’s [12] theory in order to derive the superlinear convergence of a large class of quasi-Newton updates including Broyden’s one [7]. For polyhedral convex sets we may characterize semistability: it reduces to the condition that the solution $\bar{x}$ is an isolated solution of the variational inequality linearized at $\bar{x}$. An equivalent condition is the “strong positivity condition” of Reinoza [18]. We also check that for nondifferentiable data the theory can be extended using point-based approximations (reminiscent of those of Robinson [23]) that play the role of a linearized function.

The second part of this paper is devoted to a special class of variational inequalities generalizing optimality systems. The unknowns here are couples $(x, \lambda)$ and we try to obtain conditions related to the superlinear convergence of $\{x^k\}$ alone. Indeed, we give a characterization of the superlinear convergence of $\{x^k\}$, valid under a second-order hypothesis satisfied by optimality systems for which the weak second-order sufficiency condition holds. This allows us to extend to