LEADING COEFFICIENTS OF THE EIGENVALUES OF PERTURBED ANALYTIC MATRIX FUNCTIONS

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This note contains some supplements to our earlier notes [LN II], [LN III], where the Newton diagram was used in order to obtain in a straightforward way information about the perturbed eigenvalues of an analytic and analytically perturbed matrix function.

Let \( A(\lambda) \) be an \( n \times n \)-matrix function, which is analytic in a neighbourhood of \( \lambda_0 = 0 \) with \( \det A(\lambda) \neq 0 \), and let \( \lambda_0 = 0 \) be an eigenvalue of \( A(\lambda) \), that is \( \det A(\lambda_0) = 0 \). As in [N], [LN II], [LN III], we are interested in the eigenvalues of the perturbed function \( T(\lambda, \epsilon) = A(\lambda) + B(\lambda, \epsilon) \) near \( (0, 0) \) under the assumption that \( B(\lambda, \epsilon) \) is analytic at \( (0, 0) \) and \( B(\lambda, \epsilon) = 0 \). We retain the notation of [LN II], [LN III], in particular, let \( g = \dim \ker A(0) \), let \( A(\lambda) = E(\lambda) D(\lambda) F(\lambda)^{-1} \) be a local Smith form of \( A(\lambda) \) at \( \lambda_0 \) with \( D(\lambda) = \text{diag}(\lambda^{m_1}, \ldots, \lambda^{m_n}) \), where we assume that the nonnegative integers \( m_1, \ldots, m_n \) are ordered such that

\[
0 < m_1 \leq m_2 \leq \ldots \leq m_g, \quad m_{g+1} = \ldots = m_n = 0.
\]

Denote by \( k \) the number of groups of mutually equal \( m_i \), by \( n_j \) the number of elements of the \( j \)-th group and \( \tilde{n}_j := n_1 + \ldots + n_j, \tilde{m}_j := m_{\tilde{n}_j}, j = 1, 2, \ldots k \).

In [LN II], [LN III] it was shown that the decisive role for the behavior of the eigenvalues of \( T(\lambda, \epsilon) \) near \( \lambda_0 \) for small \( \epsilon \) is played by the \( g \times g \) matrix \( H \) which is formed by the first \( g \) rows and columns of the matrix \( E(0) \frac{\partial B}{\partial \epsilon}(0, 0) F(0)^{-1} \), where \( E \) and \( F \) are given by the local Smith form of \( A(\lambda) \) at \( 0 \). Let \( H_j \) be the submatrix of \( H \) formed by the last \( n_j + \ldots + n_k = g - \tilde{n}_{j-1} \) rows and columns and denote \( \triangle_j := \det H_j \).

In this note we always assume that

\[
\triangle_j \triangle_{j+1} \neq 0
\] (1)

holds for some \( j \in \{1, \ldots, k\} \). Then, according to Corollary 1 in [LN III], there are exactly \( n_j \tilde{m}_j \) eigenvalues \( \lambda_{\nu\sigma}(\epsilon), \nu = 1, \ldots, n_j, \sigma = 1, \ldots, \tilde{m}_j \), of \( T(\lambda, \epsilon) \) near \( \lambda_0 \) with leading exponent \( 1/\tilde{m}_j \), more precisely

\[
\lambda_{\nu\sigma}(\epsilon) = \gamma_{\nu, \sigma} \Theta_{\tilde{m}_j, \sigma} e^{1/\tilde{m}_j} + o(|\epsilon|^{1/\tilde{m}_j} \lambda_0)(\epsilon \to 0), \quad \nu = 1, \ldots, n_j, \sigma = 1, \ldots, \tilde{m}_j.
\] (2)
We call the number $\alpha_{\nu\sigma} := \gamma_\nu \Theta_{n_j, \sigma}$ the leading coefficient (l.c.) of the Puiseux expansion of $\lambda_{\nu\sigma}(\varepsilon)$. Here $\gamma_\nu$ has the property that $\xi_\nu = (\gamma_\nu)^{n_j}$ is a root of the determining equation

$$\Delta_j + \sum_{i=1}^{n_j-1} \Delta_{ij} \xi^i + \Delta_{j+1} \xi^{n_j} = 0$$

(comp. e.g. (5) in [LN III]). Conversely, to each root $\xi_\nu$ of (3) there are $n_j$ l.c.'s $\alpha_{\nu\sigma}$ given by all the $m$-th roots of $\xi_\nu$. We refer to [LN III] for the definition of $\Delta_{ij}$.

If we observe the definition of $\Delta_j, \Delta_{ji}$ and the Lemma of [LN III] it follows that the equation (3) can be written as

$$\det(H_j + \xi P_j) = 0$$

with

$$P_j := \text{diag}(1, \ldots, 1, 0, \ldots, 0) \text{ if } j < k, \ P_k := I.$$ 

Thus, in order to find the $\alpha_{\nu\sigma}$ the equation (4) has to be solved. We write the matrices in (4) in block form:

$$P_j = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ H_j = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with an $n_j \times n_j$-matrix $A$ and $D = H_{j+1}$. Observing that (1) means that $H_j$ and $D$ are nonsingular, it follows

$$H_j + \xi P_j = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1} C + \xi I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1} C & I \end{pmatrix}.$$ 

Therefore the equation (4) reduces to the equation

$$\det(A - BD^{-1} C + \xi I) = 0.$$ 

In the rest of this note we assume that $H_j$ is hermitian, that is $A = A^*, H_{j+1} = H_{j+1}^* (= D = D^*)$ and $C = B^*$. If we put $\eta = -\xi$, then (7) becomes the characteristic equation

$$\det(A - BD^{-1} B^* - \eta I) = 0.$$ 

For a hermitian matrix $X$ by $\text{sign} X := (\kappa_+(X), \kappa_0(X), \kappa_-(X))$ we denote its signature, that is $\kappa_+(X) (\kappa_-(X))$ is the total multiplicity of its positive (negative) eigenvalues, $\kappa_0(X)$ the dimension of its kernel.

**PROPOSITION 1.** Assume that (1) holds and that $H_j$ is hermitian. Then the roots of (8) are real and nonzero, $\kappa_+(H_j) - \kappa_+(D)$ of them being positive, $\kappa_-(H_j) - \kappa_-(D)$ being negative.