Nearly One Dimensional Singularities of Solutions to the Navier–Stokes Inequality

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Abstract. There exists a function $u$ satisfying (1) $u$ is a weak solution to the Navier–Stokes equations of incompressible fluid flow in three-space with an external force that reduces the speed at every point, (2) the internal singularities of $u$ have Hausdorff dimension close to 1.

Section 1. Introduction

The theorem below is an improvement on Theorem 1.1 of [4]. The difference between the two theorems is in the size of the singular set $S$. This set consisted of only one point in [4]. In the present paper, the Hausdorff dimension of $S$ is nearly one. In what follows, the laplacian $\Delta$ and the gradient $\nabla$ of a function defined on a subset of $\mathbb{R}^3 \times \mathbb{R}$ will involve only the $\mathbb{R}^3$ variables.

Theorem. If $\zeta < 1$ then there exists a Cantor set $S \subset \mathbb{R}^3 \times \{1\}$ and there exist functions $u: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ and $p: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ satisfying the following properties:

there is a compact set $K \subset \mathbb{R}^3$ such that $u(x, t) = 0$ for all $x \notin K$, (1.1)

for fixed $t$, the function $u_t: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $u_t(x) = u(x, t)$ is a $C^\infty$ function, (1.2)

$$\sum_{i=1}^{3} (\partial u_i/\partial x_i)(x, t) = 0,$$ (1.3)

$$p(x, t) = \int_{\mathbb{R}^3} \sum_{i=1}^{3} \sum_{j=1}^{3} (\partial u_j/\partial x_i)(y, t)(\partial u_i/\partial x_j)(y, t)(4\pi|x - y|)^{-1} dy,$$ (1.4)

there exists $M < \infty$ such that $\|u_t\|_2 < M$ for all $t(u_t$ defined in (1.2)), (1.5)

$|\nabla u|^2, |u|^3$ and $|u||p|$ are integrable, (1.6)

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if \( \phi: \mathbb{R}^3 \times (0, \infty) \to \mathbb{R} \) is a \( C^\infty \) function with compact support and \( \phi \geq 0 \), then

\[
\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \phi \,dx \,dt \leq \int_0^\infty \int_{\mathbb{R}^3} (2^{-1} |u|^2 + p)u \cdot \nabla \phi + \int_0^\infty \int_{\mathbb{R}^3} 2^{-1} |u|^2 \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right),
\]

(1.7)

\( u \) is not essentially bounded on any neighborhood of any point in \( S \),

(1.8)

the Hausdorff dimension of \( S \) is greater than \( \zeta \).

(1.9)

The introduction of [4] contains a discussion of the heuristics of this type of theorem. Briefly, we are interested in solutions to

\[
\frac{\partial}{\partial t} u_i = - \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \Delta u_i + f_i,
\]

(1.10)

\[
\sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = 0, \quad \sum_{i=1}^3 \frac{\partial f_i}{\partial x_i} = 0, \quad \sum_{i=1}^3 f_i u_i \leq 0.
\]

This says that \( u \) is a solution to the Navier–Stokes equations of incompressible fluid flow with an external force \( f \) that is divergence free and pushes against the flow at every point of space-time. Properties (1.3), (1.4), (1.7) say that (1.10) is satisfied in a weak sense. The right definition of weak solution is obtained by multiplying (1.10) by a nonnegative test function \( \phi \) and integrating.

It can be shown that the theorem in this paper is nearly optimal. The proof of Theorem 2.1 of [3] implies that the intersection of the singular set with any hyperplane of the form \( \mathbb{R}^3 \times \{t\} \) cannot have dimension greater than 1. Therefore, the conditions stated in the theorem (which include \( S \subset \mathbb{R}^3 \times \{1\} \)) always force \( \dim(S) \leq 1 \). Furthermore, L. Caffarelli, R. Kohn and L. Nirenberg proved, in a slightly different context, that it is not possible to increase the dimension of \( S \) beyond 1 by relaxing the requirement \( S \subset \mathbb{R}^3 \times \{1\} \). This result appears in [1].

**Section 2. Preliminaries**

We recall some of the notation of [4]. The set of \( C^\infty \) functions with compact support from \( U \) into \( \mathbb{R}^n \) will be denoted by \( C^\infty_c(U, \mathbb{R}^n) \). The support of a function \( f \) will be written \( \text{spt}(f) \). We set \( P = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \). If \( c = (c_1, c_2) \in \mathbb{R}^2 \) and \( c_1^2 + c_2^2 = 1 \), then \( R_c: \mathbb{R}^3 \to \mathbb{R}^3 \) is the rotation

\[
R_c(x_1, x_2, x_3) = (x_1, c_1 x_2 - c_2 x_3, c_1 x_3 + c_2 x_2)
\]

about the \( x_1 \) axis. If \( f \in C^\infty_c(P, \mathbb{R}), \) \( v = (v_1, v_2) \in C^\infty_c(P, \mathbb{R}^2), \) \( f \geq 0 \) and \( f(x) > |v(x)| \) holds for all \( x \in \text{spt}(v) \), then \( u[v, f] \in C^\infty_c(R^3, R^3) \) is defined by

\[
u[v, f](x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2), (f(x_1, x_2))^2 - |v(x_1, x_2)|^2)^{1/2}
\]

if \((x_1, x_2) \in P, \)

\[
u[v, f](R_c(x_1, x_2, 0)) = R_c(u[v, f](x_1, x_2, 0)) \quad \text{if } \quad c \in \mathbb{R}^2, |c| = 1, \quad (x_1, x_2) \in P,
\]

and \( u[v, f](x_1, 0, 0) = 0 \). Under the same hypotheses, we define \( p^*[v, f]: \mathbb{R}^3 \to \mathbb{R} \)