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On the Symplectic Structure of General Relativity

Abhay Ashtekar[1,2†] and Anne Magnon-Ashtekar[3§]

1 Physics Department, Syracuse University, Syracuse, NY 13210, USA*
2 Département de Physique, Université de Clermont-Fd., F-63170 Aubière, France
3 Département de Mathématiques, Université de Clermont-Fd., F-63170 Aubière, France

Abstract. The relation between the symplectic structures on the canonical and radiative phase spaces of general relativity is exhibited.

1. Introduction

There are available in the literature, two Hamiltonian descriptions of general relativity. The first and the more established one is based on spacelike hypersurfaces and uses the initial value formulation of general relativity and the Dirac theory of constrained systems [1, 2]. Over the years, this formulation has been systematically developed and refined by several authors and has shed considerable light on the structure of Einstein's theory. (See, e.g., [3].) In particular, these investigations have brought out the role of the Arnowitt–Deser–Misner [4] energy-momentum as the generator of space-time translations [5] and have paved the way for canonical quantization of gravity [3]. The second Hamiltonian description became available more recently [6]. It is based on null infinity [7] and uses techniques from the gravitational radiation theory in exact general relativity. (See especially, [8] and [9].) Here the focus is on the radiative aspects of the gravitational field; the phase space is the space of radiative modes. This description has also given one new insight. In particular, fluxes of energy-momentum and angular momentum carried away by gravitational waves have been shown to be the generators of the Bondi–Metzner–Sachs (BMS) group, the asymptotic symmetry group at null infinity [10]. More importantly, the formulation has enabled one to carry out the asymptotic quantization of the non-linear gravitational field [6, 11].

In view of this situation, it is natural to ask for the relation between the two descriptions. Apart from its intrinsic interest, such an analysis would clarify several issues which arise in the two frameworks separately. For example, since the radiative

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§ Supported in part by crédits ministériels, tranche spéciale

* Permanent address

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phase space is not constructed from a cotangent bundle over a configuration space, the symplectic tensor field thereon had to be simply postulated [10]. Here, one was guided by general considerations such as the requirement that the Poisson bracket between the basic variables should have the dimensions of action, that one should obtain the correct results in the weak field limit, and that the expression of the symplectic structure should fit in the pattern suggested by the spin zero and one fields. However, one could not show that these considerations suffice to determine the symplectic tensor field uniquely. It is therefore desirable to have as strong an evidence as possible supporting the choice that was made. A strong—perhaps the strongest possible—evidence would be that the chosen symplectic structure is, in an appropriate sense, the same as the one on the canonical phase space. The canonical approach would also be enriched from the analysis of its relation to the radiative framework. For example, the canonical quantization programme has met with severe difficulties in the construction of a Hilbert space of states (or a substitute thereof). In the approach based on the radiative phase space, on the other hand, these difficulties do not arise: one can readily construct not only the Fock spaces of asymptotic gravitons but also the Hilbert spaces required to handle the infrared problems, i.e., which are analogous to the charged sectors in quantum electrodynamics [12]. Therefore, an understanding of the relation between the two phase spaces may give one considerable insight in the Hilbert space problem of canonical quantization. In particular, the analysis may shed light on the nature of the (canonical) quantum vacuum, which, one now suspects, may not be simply a gaussian peaked at the flat metric.

The purpose of this paper is to provide the first steps towards establishing the relation between the two phase spaces. At an intuitive level, one may divide the problem into two parts: differential geometric issues and functional analytic difficulties. In a broad sense, this paper resolves the first part. More precisely, we shall assume that globally hyperbolic, vacuum, asymptotically flat, horizon-free space-times exist and show that each such space-time leads to a natural *symplectic structure preserving* identification of a point of the canonical phase space with a point of the radiative phase space.

The main obstacle in relating the two phase spaces is, of course, that whereas the canonical phase space is constructed from initial data sets on *space-like* surfaces, the radiative phase space consists of certain equivalence classes of connections on null infinity, $\mathscr{I}$.[1] Therefore, to exhibit the relation between the two, we shall introduce a structure which can interpolate between the two regimes: the symplectic vector space of *linearized* gravitational fields on a globally hyperbolic asymptotically flat, vacuum space-time without horizons. This introduction serves the following purpose. A linearized solution induces on any Cauchy surface a set of linearized Cauchy data and may be therefore regarded as a tangent vector at a point of the canonical phase space. As one might expect, this identification preserves the symplectic structure: We shall show explicitly that the symplectic structure on the space of linearized solutions (off a fixed background) reduces to the symplectic structure evaluated at the tangent space of any point of the canonical phase space

1 Throughout this paper, the symbol $\mathscr{I}$ will stand for future *or* past null infinity