Borel Summability of the $1/N$ Expansion for the $N$-Vector $[O(N)$ Non-Linear $\sigma]$ Models

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Abstract. We construct an analytic interpolation in $1/N$ for the $N$-vector $[O(N)$ non-linear $\sigma]$ models with $N$-component fields on a lattice. This interpolation, valid at sufficiently high temperatures, extends over a large domain in the complex plane containing the half plane $\text{Re}(1/N) > 0$. We use this result to show that the $1/N$ expansion of the free energy density and of the correlation functions is Borel summable in the thermodynamic limit and at high temperature.

1. Introduction, Notations and Main Results

In this paper we continue a mathematically rigorous analysis of the $1/N$ expansion in the $N$-vector models, initiated by A. Kupiainen $[1,2]$. Kupiainen has shown that the $1/N$ expansion is asymptotic for two families of models, the $N$-vector models on a simple, (hyper) cubic lattice $\mathbb{Z}^d$, $d = 2, 3, 4, \ldots$, at temperatures above the critical temperature of the spherical model ($N = \infty$), and a class of weakly coupled $N$-component $\lambda|\phi|^4$ models in two space-time dimensions. A careful analysis of the $1/N$ expansion for the three-dimensional $O(N)$ $\sigma$-models in the continuum limit has been carried out by I. Aref'eva $[3]$ who, however, has not determined its nature. For a summary of the history of $1/N$ expansions and references to important, earlier work, see Kupiainen's papers $[1, 2]$.

A natural problem is to study the analyticity properties in $1/N$ and to determine the summability properties of the $1/N$ expansion for the models mentioned above. Billionnet and Renouard have recently proven that the $1/N$ expansion for weakly coupled $N$-component $\lambda|\phi|^4$ models in two dimensions is Borel-summable $[4]$. In this paper we establish the same result for the $O(N)$ non-linear $\sigma$-models on a lattice of arbitrary dimension, at high temperature. The methods used in this paper are different from the ones in $[4]$. In $[4]$ the main technical difficulty appears in the construction of the continuum (ultraviolet) limit. Here we do not construct...
the continuum limit. However, the $1/N$ expansion for the $N$-vector models is an expansion around a stationary point of a complex measure which has only "poor fall-off at infinity" and which is hard to control when $1/N$ is close to the imaginary axis. We resolve this difficulty by superimposing a second expansion related to a standard high temperature expansion. Each term in this double expansion can be calculated explicitly and turns out to be analytic in $1/N$ everywhere except on the interval $[-1/2,0]$. Our double expansion can be interpreted as a random walk—or polymer representation of thermodynamic and correlation functions of the $N$-vector models. It is inspired by work of Symanzik [5] and a rigorous version thereof [6] which has recently been applied intensively to the study of the Ising ($N = 1$) and classical rotor ($N = 2$) models [6–8]. The representation described in [6] can be interpreted as a double expansion in powers of $N$ and $\beta$. It is used in [9] to derive joint analyticity in $N$ and $\beta$ near $N = 0$, $\beta = 0$. As in [6], our double expansion represents the $N$-vector models as gases of random walks or polymer chains with soft core interaction. In contrast to the random walks used in [6], the random walks introduced in this paper make steps of arbitrary length, but with exponentially decaying probability. Our random walk representation is chosen in such a way that the spherical model limit, $N \to \infty$, is reached smoothly along rays in the half plane $\text{Re} 1/N > 0$.

The convergence of our double expansion is controlled by cluster (or polymer) expansion methods described in [10]; see also [11, 12].

We establish convergence at high temperature in a large domain of the $1/N$ plane, uniformly in the volume cutoff. We also derive bounds on the $k$th derivative in $1/N$ of thermodynamic and correlation functions. When combined with a theorem due to Nevanlinna and Sokal [13] they suffice to prove Borel summability of the $1/N$ expansion.

Our paper is organized as follows:

In the remainder of Sect. 1 we define the lattice $N$-vector models, introduce some notation and summarize our main results in the form of a theorem.

In Sect. 2 we derive the basic random walk representation of our models and express it in a compact form which makes the methods of [10, 12] accessible.

In Sect. 3 we recall the cluster expansion [10–12] and use it to control the convergence of our random walk representation. We construct the thermodynamic limit of our models and verify analyticity in $1/N$ in a large domain and prove Borel summability at $1/N = 0$, for sufficiently high temperatures; (see Theorem A below).

In Sect. 4 we discuss our results and describe some open problems related to the main theme of this paper.

Some technical estimates needed in Sects. 2 and 3 are proven in an appendix.

Throughout this paper we follow quite closely the notation introduced in [1]. We adopt the conventions that

- an empty sum $= 0$,
- an empty product $= 1$.

With each site, $j$, of the simple (hyper) cubic lattice $\mathbb{Z}^d$, $d = 2, 3, 4, \ldots$, we associate