Remarks on a Paper by J. T. Beale, T. Kato, and A. Majda

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Abstract. We prove that the maximum norm of the deformation tensor controls the possible breakdown of smooth solutions for the 3-dimensional Euler equations. More precisely, the loss of regularity in a local smooth solution of the Euler equations implies the growth without bound of the deformation tensor as the critical time approaches; equivalently, if the deformation tensor remains bounded the existence of a smooth solution is guaranteed.

The motion of an ideal incompressible fluid is described by a system of partial differential equations known as Euler equations. In [1] J. T. Beale, T. Kato, and A. Majda have given a mathematically rigorous connection between the accumulation of vorticity and the development of singularities for the three-dimensional Euler equations. In fact, they have shown that the maximum norm of the vorticity alone controls the breakdown of smooth solution of these equations. Thus one may ask: Does the blow up of the solution imply also the blow up of the deformation tensor in the maximum norm? or, may it stay bounded for a longer time? In this note we answer these questions. More precisely, we obtain the same results as those in [1], when the vorticity is substituted by the deformation tensor.

Thus we consider the system

\begin{align}
(a) & \quad u_t^k + u^j \cdot \partial_j u^k + \partial_k p = 0 \quad k = 1, 2, 3 \\
(b) & \quad \text{div } u = 0
\end{align}

where \( x \in \mathbb{R}^3, t > 0, u = u(x, t) = (u^1, u^2, u^3) \) is the velocity field, and \( p = p(x, t) \) is the pressure.

For this system the following local existence theorem is known: Given an initial velocity \( u_0 \in H^s, s \text{ integer, } s \geq 3 \) and \( \text{div } u_0 = 0 \), there exists \( T_0 = T_0(\|u_0\|_s) \) so that the system \((1)\) has a unique solution \( u \in C([0, T]: H^s) \cap C^1([0, T]: H^{s-1}) \) at least for \( T = T_0 \). (See reference in [1]).

Here we denote by \( H^s = H^s(\mathbb{R}^3) \) \( (s \text{ being a positive integer) the Sobolev space consisting of functions whose distributional derivatives up to order } s \text{ belong to } L^2(\mathbb{R}^3), \) and by \( \|u\|_s \) the norm of \( u \) in \( H^s \). Also, we use \( \omega = \nabla \times u \) for the vorticity and \( T = (T_{ij}) \ i, j = 1, 2, 3, \) where \( T_{ij} = \partial_j u^i + \partial_i u^j \) for the deformation tensor.
Theorem 1. Let $u \in C([0, T_1]; H^s) \cap C^1([0, T_1]; H^{s-1})$ be a solution of (1). Then the inequality
\[ \|u(t)\|_s \leq \|u(0)\|_s e^{C_\epsilon \int_0^\tau |T_{ij}|(\tau)\,d\tau} \] (2)
holds for all $t \in [0, T_1].$

Corollary 1. If the solution of (1) considered above exists in the time interval $[0, T_2)$ and cannot be extended beyond $T_2,$ then
\[ \int_0^{T_2} |T_{ij}|(\tau)\,d\tau = \infty \]
and, in particular,
\[ \limsup_{t \to T_2} |T_{ij}|(t) = \infty. \]

Corollary 2. If the solution of (1) considered above exists in the time interval $[0, T_3],$ and for some $T_4 > T_3$ we have that
\[ \int_0^{T_4} |T_{ij}|(\tau)\,d\tau < \infty, \]
then the solution can be extended to the interval $[0, T_4],$ in which it remains of the same type. 

Corollary 1 and Corollary 2 are immediate consequences of the local existence theorem and the estimate (2), and their proof will be omitted here. Using classical energy estimates (see [1]) one can obtain the inequality
\[ \|u(t)\|_s \leq \|u(0)\|_s e^{C_\epsilon \int_0^\tau |\nabla u_j| \cdot \omega \,d\tau} \]
for all $t \in [0, T],$ where the solution of the type considered above exists. In [1] further estimates which involve the vorticity equation and the Biot-Savart law were used to find a bound of $|\nabla u_j(t)|$ as a function of $|\nabla \omega |(t).$ Here our method of proof of (2) is based only on a careful computation of the energy estimates.

Proof of Theorem 1. In the proof of this theorem we will use the following: If $u$ is a solution of (1) and $v^1, v^2, v^3, w \in H^1(\mathbb{R}^3),$ then
\[ \int u^i \partial_j v^k \cdot v^k = 0 \quad \text{and} \quad \int u^i \cdot \partial_j w = 0. \]

These facts follow from Eq. (1) (b) and integration by parts.

First we provide the proof for the case $s = 3.$ By the above $\|u(t)\|_2 = \|u(0)\|_2$ for all $t \in [0, T_1].$ Taking $\partial^3_{lim}$ derivatives of Eqs. (1) (a), multiplying the result by $\partial^3_{lim} u^k,$ adding in $k, i, l, m$ and integrating, we obtain that
\[ \frac{1}{2} \frac{d}{dt} \|\partial^3_{lim} u^k(t)\|_2^2 + \int \partial^3_{lim} (u^i \partial_j u^k) \cdot \partial^3_{lim} u^k = 0. \] (3)