Some Considerations on the Nonlinear Stability of Stationary Planar Euler Flows*

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Abstract. We give sufficient conditions for the nonlinear stability of possibly nonsmooth stationary solutions of the two-dimensional Euler equation in symmetric bounded domains. We use, as Lyapunov functions, first integrals due to the symmetry of the problem. Moreover, we investigate the stability of smooth solutions under perturbations of the boundary. The last result is based on a generalization of the well known Arnold approach.

1.

Some years ago Arnold [1] proposed an approach to investigate the nonlinear stability of stationary Euler flows. According to the theory of finite dimensional Hamiltonian systems, the basic idea was to look for conditions ensuring the vanishing of the first variation of the energy functional and the positivity of its second variation. We briefly review the argument. For a more complete analysis we address the reader to [2], where other infinite-dimensional situations are also discussed.

Consider an incompressible ideal fluid contained in a domain $D$ bounded by two smooth curves $C_1$ and $C_0$, which are the internal and external boundary respectively. Then the following functional

$$\tilde{H} = \frac{1}{2} \int_D u^2 \, dx \, dy + \int_D \Phi(\omega) \, dx \, dy + \sum_{i=0}^1 a_i \int_{\mathcal{C}_i} u \cdot d\ell$$

(1.1)

($u$ is the velocity field, $\omega = \text{curl} u = \partial_x u^{(2)} - \partial_y u^{(1)}$, $\Phi$ a real valued function, $a_i$ real numbers) is a constant of motion, each of the three terms appearing in the right-hand side of (1.1) being first integrals.

The condition $\delta \tilde{H}(\bar{u}) = 0$, for $\bar{u}$ stationary solution, yields (see [2] for details)

$$a_i = -\Phi'(\bar{\omega}) \quad (\bar{\omega} = \text{curl} \bar{u})$$

(1.2)

$$\bar{u} = \mathcal{V} \Phi'(\bar{\omega})$$

(1.3)

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(We notice that $\tilde{\omega}$ is constant on $\mathcal{C}_i$ because $\tilde{u}$ is stationary and hence $\tilde{u} \perp V \tilde{\omega}$.)

Equation (1.3) can be solved and we obtain, for the solution $\Phi$,

$$\Phi''(\tilde{\omega}) = \frac{\bar{V} \bar{\Psi}}{\bar{V} \tilde{\omega}} = \frac{\tilde{u}}{\bar{V} \tilde{\omega}},$$

(1.4)

where $\bar{\Psi}$ is the stream function (i.e., $\overline{u} = \bar{V} \bar{\Psi}$) and the above ratios are well defined by virtue of the collinearity of $\overline{u}$ and $\bar{V} \tilde{\omega}$ for stationary solutions.

The second variation is

$$(\delta^2 \tilde{H})(\bar{u}) = \int_D (\delta \bar{u})^2 dx dy + \int_D \Phi''(\bar{\omega})(\delta \bar{\omega})^2 dx dy,$$

(1.5)

which is definite positive if $\Phi'' > 0$. In this case we have a natural norm that is $\int_D (u^2 + \omega^2)$, by means of which we investigate the stability problem.

Actually one can prove ([1, (1969)], [2], and also Sect. 3 of the present paper where this argument is applied) the nonlinear stability, in such a form, of stationary solutions for which $C_2 \geq \Phi'' \geq C_1 > 0$.

In the presence of symmetry other first integrals are available. For instance in the case of a plane periodic flow between two parallel plates, the following functional

$$J = \int_D \gamma \omega dx dy$$

(1.6)

is a first integral. It is natural to investigate how the above variational argument works for this functional.

Defining

$$\hat{J} = J + \int \Phi(\omega) dx dy,$$

(1.7)

the condition

$$\delta \hat{J}(\bar{u}) = 0$$

(1.8)

implies

$$- \Phi'(\bar{\omega}) = \gamma, \quad - \Phi''(\bar{\omega}) = (\partial_y \bar{\omega})^{-1}$$

(1.9)

for $\bar{\omega} = \bar{\omega}(x, y)$ depending only on $y$.

We assume the condition

$$C_2 \geq - (\partial_y \bar{\omega})^{-1} \geq C_1 > 0,$$

(1.10)

and extend the function $\Phi$ out of the range of $\bar{\omega}$ to a smooth function, again denoted by $\Phi$, satisfying the condition $C_1^{-1} \geq \Phi'' \geq C_2^{-1}$. Then the second variation

$$\delta^2 \hat{J} = - \int (\partial_y \bar{\omega})^{-1} (\delta \omega)^2 dx dy$$

(1.11)

is positive. For a perturbation $\omega_i$ of the equilibrium, we have

$$\hat{J}(\omega_i) - \hat{J}(\bar{\omega}) = \int_D [y(\omega - \bar{\omega}) + \int_D [\Phi(\omega) - \Phi(\bar{\omega})]]$$

$$= \int_D [y(\omega - \bar{\omega}) + \Phi'(\bar{\omega})(\omega - \bar{\omega})]$$

$$+ \int_D [\Phi(\omega) - \Phi(\bar{\omega}) - \Phi'(\bar{\omega})(\omega - \bar{\omega})].$$

(1.12)