ASYMPTOTICS OF SPECTRUM UNDER INFINITESIMALLY FORM-BOUNDED PERTURBATION

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Let $A \geq 1$ be a selfadjoint operator with discrete spectrum and known distribution function of its spectrum $N(r, A)$. Suppose $B$ is a (nonselfadjoint) operator that is form-bounded with respect to $A$ with relative bound zero. If in addition $\lim_{r \to \infty} N(r + \epsilon r, A)N(r, A)^{-1} = 1$ then $N(r, A + B) = N(r, A)(1 + o(1))$, where $A + B$ is the operator defined as form sum. The applications to the Schrödinger operator with polynomially growing potential and to the third boundary value problem for the second order elliptic operator are given.

INTRODUCTION

Let $A$ be a selfadjoint operator with discrete spectrum and known distribution function of its spectrum

$$N(r, A) = \sum_{|\lambda_j(A)| \leq r} 1.$$ 

A.S. Markus and V.I. Matsaev [1] obtained the following result.

THEOREM 0. Suppose $B$ is a (nonselfadjoint) operator relatively compact with respect to $A$ (that is the operator $B(A + i)^{-1}$ is compact). Then for each $\delta > 0$ there exists $r(\delta) > 0$ such that for $r > r(\delta)$

$$|N(r, A + B) - N(r, A)| \leq C(N(r + \delta r, A) - N(r - \delta r, A))$$

where the constant $C$ is independent of $r$ and $\delta$. If in addition

$$\lim_{\delta \to 0} N(r + \delta r, A)N(r, A)^{-1} = 1$$

then

$$N(r, A + B) = N(r, A)(1 + o(1)).$$

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In Section 1 in the case of positive $A$ we give a generalization of Theorem 0 where $B$ is supposed to be infinitesimally form-bounded with respect to $A$ (Theorems 1 and 2). In Section 2 we apply this result to the nonselfadjoint perturbations of the Schrödinger operator with polynomially growing potential and to the third boundary value problem for the second order elliptic operator (Examples 1, 2 and 3).

The case when $B$ is "$p$-subordinated" to $A$ was studied in [1, Theorem 2.1]. The generalization for the form $p$-subordination case was given in [2]. The detailed discussion can be also found in [18] and [9, Section 20]. We consider the restrictions on $B$ in the terms of form-boundedness with respect to $A$ with relative bound zero. It enables us to follow essentially the proof of the operator $p$-subordination case [1, Theorem 2.1].

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1. THE MAIN RESULT

Let $H$ be a separable Hilbert space and denote $|| \cdot ||$ and $\langle \cdot , \cdot \rangle$ to be the norm and the scalar product in $H$. Let $A$ be a positive selfadjoint operator in $H$, $A \geq 1$. Suppose $A$ has purely discrete spectrum $\sigma(A)$ ($A^{-1}$ is compact) and known distribution function of the spectrum $N(r, A)$. Denote $Q(A)$ to be the domain of the sesquilinear form $A[u] = \langle A^{1/2}u, A^{1/2}u \rangle$. Suppose the sesquilinear form $B[u]$ is form-bounded with respect to $A$ with relative bound zero. That is $Q(B) \subset Q(A)$ and for each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for all $u \in Q(A)$

$$|B[u]| \leq \varepsilon A[u] + c(\varepsilon)||u||^2. \quad (1)$$

Denote $A + B$ to be the $m$-sectorial operator associated with the form sum $A[u] + B[u]$ [5, Chapter 6]. The operator $A + B$ also has purely discrete spectrum [5, Chapter 6, Theorem 3.4]. For $\lambda \notin \sigma(A+B)$ denote $R_\lambda(A+B) = (A+B - \lambda)^{-1}$.

**THEOREM 1.** Suppose the condition (1) holds. Then for each $\delta \in (0,1)$ there exists $r_0(\delta) > 0$ such that for all $r > r_0(\delta)$

$$|N(r, A+B) - N(r, A)| \leq D(N(r + \delta r, A) - N(r - \delta r, A)) \quad (2)$$

where the constant $D$ is independent of $r$ and $\delta$.

**THEOREM 2.** Suppose the condition (1) holds. If in addition

$$\lim_{r \to \infty} \frac{N(r + \delta r, A)}{N(r, A)} = 1 \quad (3)$$

then

$$N(r, A+B) = N(r, A)(1 + o(1)). \quad (4)$$