The Stability of Rotating Vortex Patches*

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* Dedicated to the memory of my grandmother, Lang-Chang Lee Wan

Abstract. In this paper we examine the nonlinear and linear stability of various rotating vortex patches. These patches include the Kirchhoff ellipse, the Kelvin waves, and the co-rotating uniform m vortices. These are achieved by using relative variational methods and spectral analysis. Thus, we extend Arnol'd's idea for stability problems in [1965, 1969] to a non-smooth symmetric setting and also relate that to the usual linear stability analysis.

1. Introduction

Consider the motion of an incompressible flow with unit density in $\mathbb{R}^2$ in the absence of external forces. At any instant, the velocity field $(u, v) = (\psi_y, -\psi_x)$ for some stream function $\psi$ on $\mathbb{R}^2 = \{x = (\xi, \eta)\}$. The vorticity $\omega = \psi_x - u\psi = -\psi_x - \psi_y = -\Delta \psi$. We like to use the vorticity $\omega$ as the independent variable. Given $\omega$, let us choose a stream function $\psi = \int G_{\omega}(1/2\pi)\ln(1/|\xi - \bar{x}|)dx'$, so that the velocity field is zero at infinity. The vorticity evolves according to the vorticity equation: $\omega_t + u\omega_x + v\omega_y = 0$. Denote by $\Phi_t(\omega)$ the vorticity at time $t$, with initial vorticity $\omega$.

The energy $E$, the circulation $C$, the centre $(\bar{x}_0, \bar{y}_0)$, and the angular momentum $J$ are preserved under the motion $\Phi_t$. Recall that for a given vorticity $\omega$, $E = \frac{1}{2}\langle \omega, \psi \rangle = \frac{1}{2}\int_{R^2} \omega(\xi)\psi(\xi)d\xi$, $C = \int_{R^2} \omega(\xi)d\xi$, $\bar{x}_0 = \int_{R^2} \bar{x}\omega(\xi)d\xi$, $\bar{y}_0 = \int_{R^2} \bar{y}\omega(\xi)d\xi$, and $J = \int_{R^2} |\bar{x}|^2 \omega(\bar{x})d\bar{x}$. A vortex patch $\omega$ is a vorticity in the form $\chi_A$, where $\chi_A$ stands for the characteristic function for a bound (measurable) set $A$ in $\mathbb{R}^2$. $\chi_{A_j}$ is called a component of $\chi_A$, if $A_j$ is a component of $A$. Vortex patches and their components are all preserved under the motion $\Phi_t$.

A vortex patch $\chi_A$ is said to be stationary if $\Phi_t(\chi_A) = \chi_A$ for all $t \geq 0$. A vortex patch is said to be rotating if $\Phi_t(\chi_A) = \chi_{R_{\theta}A}$ for all $t \geq 0$, where $R_{\theta}$ stands for a rotation through angle $\theta$. The Kirchhoff vortices $\chi_E$ (see Sect. 4) in which $E$ is an ellipse, are our model for rotating vortex patches. Two families of rotating vortex patches have been found recently. They are (a) the m-fold symmetric "Kelvin" waves...
\( \mathcal{H}_m \) (see Deem and Zabusky [8], Burbea and Landau [7], Wu, Overman and Zabusky [33], etc), (b) the co-rotating uniform \( m \) vortices (see Dritschel [9]). Denote by \( \mathcal{V} \) the space of all vortex patches with a \( L^1 \)-topology induced from the \( L^1 \)-norm on vorticities. Throughout this paper, our stationary or rotating patches will have \( C^1 \) boundaries.

Our basic problem in this paper is to examine the stability of rotating vortex patches from various viewpoints. More precisely, we aim to carry out theoretical investigations and concrete computations of the following topics:

(a) the \( L^1 \)-stability via an energy method,
(b) the neutral stability (for a definition, see Sect. 3) through a spectral analysis,
(c) the relationship between the energy method and the spectral analysis.

Let us mention briefly (some of) of the literature which is closely related to our studies. Some general references are Lamb [17], Aref [2] and Zabusky [34]. In 1880, Kelvin ([17] p. 230) established the neutral stability of the circular patch in the plane. In 1887, Kelvin [15] proposed a variational principle for steady vortex motions. In particular, a steady vortex motion is stable if the energy reaches a maximum or minimum within a given vorticity and given moment of momentum at that vortex motion. Arnol'd [3, 4] presented a method for proving a nonlinear version of the classical Rayleigh criterion for neutral stability of 2-dim shear flows. It involves a combination of a relative variational principle and convexity arguments.


In 1876, Kirchhoff [17, 18] found his elliptic vortex. It was proved by Love [18] that Kirchhoff vortex \( \chi_K \) is neutrally stable iff its eccentricity \( \leq 2\sqrt{2}/3 \). The \( L^1 \)-stability for the same range, has just been established in Tang [26]. The rotating motion of two equal uniform vortices was studied by Saffman and Szeto [25]. The neutral stability of Kelvin waves and co-rotating uniform \( m \) vortices have been analyzed by numerical methods in [7, 10] respectively.

Now, let us outline the general lines of our approach. It is rewarding to put the motion of incompressible inviscid flows in \( \mathbb{R}^2 \) in a Hamiltonian setting. The symplectic leaf \( M \) through a vortex patch \( \omega_0 \) is an isovortical surface, consisting of all isovortical variations of \( \omega_0 \) (cf. [20]). The nature action of rigid motions in \( \mathbb{R}^2 \) leaves the leaf \( M \) (a symplectic manifold), and the energy \( E \) (a Hamiltonian) invariant. The angular momentum, centre of vorticities are the corresponding conserved quantities.

A rotating vortex patch will be regarded as a relative equilibrium with centre zero \( (\vec{x}_0 = \vec{y}_0 = 0) \) and the isotropy group = the circle group \( S^1 \). For a non-stationary rotating patch \( \chi_A \), the \( S^1 \) action is free near \( \chi_A \) and the subspace \( M_0 = \{ \omega \in M | J(\omega) = J(\chi_A), \vec{x}_0(\omega) = \vec{y}_0(\omega) = 0 \} \) is a \( S^1 \) invariant manifold. We expect that \( \chi_A \) is a critical point of the energy function on \( M_0 \). Furthermore, we can verify the nonlinear stability of this rotating patch \( \chi_A \) by establishing that the energy \( E \) has a non-