Stability of Classical Solutions of Two-Dimensional Grassmannian Models

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Abstract. We show that the only finite-action solutions of the two-dimensional Grassmannian $\sigma$-model that are stable under small fluctuations are the (anti-)instanton solutions.

0. Introduction

(0.1) The two-dimensional Grassmannian $\sigma$-model is a field theory which shares many of the properties of the (more complicated) four-dimensional non-abelian gauge theories: for instance, the action is conformally invariant, there is a topological charge and the associated (anti-)instantons minimise the action among all fields with the same charge. For a survey of this theory, see [11].

(0.2) It is of interest to know whether there exist any non-instanton solutions in this model that are stable under small fluctuations. It is the purpose of this article to answer this question in the negative; thus all non-(anti-)instanton solutions are saddle points for the action. Our technique uses methods of Algebraic Geometry to ensure a sufficiently large number of non-positive modes for the fluctuation operator so that stability is only possible for (anti-)instanton solutions. These non-positive modes are essentially provided by solutions of the background fermion problem.

1. Preliminaries

(1.1) The non-linear $\sigma$-model is a field theory where the dynamical variable takes values in a Riemannian manifold $(N, h)$. The Lagrangian density and action for this model are given by

$$L(\phi) = h_{\alpha\beta} \partial_\mu \phi^\alpha \partial^\mu \phi^\beta, \quad S = \int L d^n x.$$  \hspace{1cm} (1)

We are interested in finite-action solutions of the equations of motion, which are known to mathematicians as harmonic maps (see e.g. [3]). We shall restrict attention to the 2-dimensional Euclidean version of the model, which is of most interest to physicists since it shares a number of properties with 4d non-abelian gauge theories. In particular, in this case, the action is conformally invariant and, by a result of Sacks and Uhlenbeck [9], any finite-action solution of the equations of motion extends to a solution on the conformal compactification of $\mathbb{R}^2$, the Riemann sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Henceforth therefore, we shall suppose, without
loss of generality, that all fields are defined on $S^2$ since we may then apply the methods of Algebraic Geometry.

(1.2) From now on, we take as our manifold $(N, h)$, the complex Grassmannian $G_{r,n}$ which is the coset space $\frac{U(r) \times U(n-r)}{U(n)}$. Following Zakrzewski [11], we identify $G_{r,n}$ with the rank $r$ projection matrices, i.e., $n \times n$ matrices $\varphi$ satisfying

$$\varphi^2 = \varphi, \quad \varphi = \varphi^+, \quad \text{rank } \varphi = r,$$

where $^+$ denotes Hermitian conjugation. Differentiating (2), we see that the tangent space to $G_{r,n}$ at $\varphi$ is the set of Hermitian matrices, $A$, satisfying

$$\varphi A \varphi = 0 = (1 - \varphi) A (1 - \varphi).$$

The Lagrangian density (1) in this case is given by

$$L(\varphi) = \text{trace } \partial^\mu \varphi \partial^\nu \varphi,$$

while the equations of motion are

$$[\varphi, \partial^\mu \partial^\nu \varphi] = 0,$$

together with the constraint (2).

(1.3) $G_{r,n}$ is a Kähler manifold with Kähler 2-form $\omega$, so that a field has a topological charge with density

$$q = i \epsilon_{\mu \nu} \text{trace } [\partial^\mu \varphi, \varphi] \partial^\nu \varphi, \quad Q = \int d^2 x q = \int_{S^2} \varphi \ast \omega.$$

Then $S \geq |Q|$ with equality for the (anti-)instanton solutions of (5) which thus minimise the action over all fields with the same charge. If we replace our Euclidean co-ordinates $(x_1, x_2)$ by holomorphic co-ordinates $x_+ = x_1 + i x_2$, then the instanton, respectively anti-instanton, equations are given by (a), respectively (b), below:

(a) $\varphi \partial_+ \varphi = 0$, \quad (b) $\varphi \partial_- \varphi = 0$.

The topological charge admits a geometrical interpretation which will be useful below: a field $\varphi$ defines vector bundles $\varphi$, $\varphi^\perp$ over $S^2$ of ranks $r$ and $n-r$ respectively, by

$$\varphi_\ast = \text{Image } \varphi(x) \subset \mathbb{C}^n, \quad \varphi^\perp_\ast = \text{Kernel } \varphi(x) \subset \mathbb{C}^n.$$

Clearly, $\varphi \oplus \varphi^\perp$ provides a non-trivial splitting of the trivial bundle $S^2 \times \mathbb{C}^n$. Suitably normalising the volume of the Riemann sphere, we have

$$Q = - \deg(\varphi),$$

where $\deg(\varphi)$ denotes the first Chern class of $\varphi$ evaluated on the generator of $H_2(S^2)$.

(1.4) Now let $\varphi$ be a finite action solution of the equations of motion (5) and let $\varphi_t$ be a small fluctuation about $\varphi = \varphi_0$ with $\left. \frac{\partial \varphi_t}{\partial t} \right|_{t=0} = B$. Clearly, $B$ satisfies (3) and,