ON THE FOUR BLOCK PROBLEM, II: THE SINGULAR SYSTEM

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In this paper we study the singular values of a "four block operator" which naturally appears in control engineering. This extends our previous results from [7]. As before, we will show that this question is reducible to a skew Toeplitz operator problem of the kind studied in [1].

1. INTRODUCTION

This paper is the sequel to [7] in which we continue our study of the spectral properties of certain "four block operators" from control and systems engineering. These operators also have a number of intriguing mathematical properties in the sense that they are natural extensions of both the Hankel and Toeplitz operators. For this reason they fit into the skew Toeplitz framework developed in [1]. See also the monograph of Francis [10], the lecture notes of Doyle [3], and the references therein for more details about this research area.

We now will give a precise mathematical statement of the four block problem. Let $w, f, g, h, m \in H^\infty$, where $w, f, g, h$ are rational and $m$ is nonconstant inner. (All of our Hardy spaces will be defined on the unit disc $D$ in the standard way.)

Set

$$\mu := \inf \{\| \begin{bmatrix} w & f \\ g & h \end{bmatrix} q : q \in H^\infty\}$$

where for a matrix
with $a, b, c, d \in L^\infty$, we set
\[
\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \|_{\infty} := \operatorname{ess} \sup \left\{ \| \begin{bmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta) \end{bmatrix} \| : |\zeta| = 1 \right\}.
\]
(The norm of each of the above 2x2 matrices for $\zeta \in \partial D$ is taken as a linear operator on $C^2$.) Then the four block problem amounts to calculating the quantity $\mu$. Note that for $f = g = h = 0$, this reduces to the classical Nehari problem. In this paper, we will identify $\mu$ as the norm of a certain four block operator (see Section 2 for the precise definition), and then in Sections 3 and 4 give an explicit determinantal formula for its computation.

The techniques given here are based on the previous work in [1], [6], [7], [8], [9], [15], [16]. In particular, our work in the first part of this series [7] gives an easily computable upper bound for $\mu$ which can be utilized in a simple iterative procedure for the actual computation of $\mu$ itself. (See [7], and Section 5 below.) It is important to note that [7] gives an elementary way of computing the norm of the "two block operator". See also the related work of [11] and [15].

Finally the skew Toeplitz framework developed in [1] also leads to the norm of four block operators associated to multivariate systems. This will be the topic of the third paper in this series in which we will apply the block determinantal formula of [1] to the specific four block structure. (See [1] for the precise definition and a more complete discussion about skew Toeplitz operators.)

We moreover plan to have an applied version of the present work in which the connections to engineering problems will be more explicitly discussed.

Once again, we would like to thank John Doyle and Bruce Francis for exciting our interest in this problem. This research was supported in part by grants from the Research Fund of Indiana University, NSF (ECS-8704047), and the AFOSR-88-0020.