GENERALIZED LIE THEOREMS AND JORDAN WHEELS

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The present article is devoted to Jordan analogs of the matrix ordinary differential equation

\[ u_t = Au + Pu + uQ + R. \]  

Here \( u(t) \) is an unknown matrix of dimension \( n \times n \), and \( A, P, Q, \) and \( R \) are constant fixed matrices. If \( A \) is nondegenerate, then by a substitution of variables of the form \( \tilde{u} = \alpha u + \beta \), where \( \alpha \) and \( \beta \) are certain constant matrices, \( A \) becomes one and \( Q \) zero. Next, the standard substitution \( v = u + \alpha u^{-1} \) reduces Eq. (1) to a matrix second-order linear function with constant coefficients (which, of course, is integrable by quadrature). The integrability of Eq. (1) with degenerate coefficient \( A \) is not, at first glance, all that obvious.

Equation (10) is a special case of quadratic systems of the form

\[ (u^i) = C^i_{jk}u^j + R^i + S^i, \quad i, j, k = 1, \ldots, N. \]  

Summation is assumed to occur over repeated indices.

In the present article we consider a special class of systems of the form (2) associated with Jordan algebras (cf. [1]). We will refer to these systems as Jordan "wheels." By means of Jordan multiplication "\( \cdot \)" the Jordan wheel is written in the following way:

\[ u_t = \{u, a, u\} + p \cdot u + \lambda(u) + r. \]

Here \( u(t) \) is a function of a real variable \( t \) that takes values in some Jordan algebra \( \mathfrak{U} \), that is finite-dimensional over \( \mathbb{C} \), \( p \) and \( r \) are the constant elements of \( \mathfrak{U} \), \( \lambda \) is its differentiation operators, and

\[ \{x, y, z\} \overset{def}{=} (x \cdot y) \cdot z + (z \cdot y) \cdot x - (x \cdot z) \cdot y. \]

It is easily verified that the system (3) coincides with the system (1) for simple Jordan algebras from the series \( A_n \).

Remark. The property of the system (2) to be a Jordan wheel is invariant relative to linear substitutions of the unknowns. It is easy to give simple necessary (and, in the case in which \( a \) is an invertible element, sufficient) conditions of the system (2) to be Jordan. For example, the constants \( C^i_{jk} \) must satisfy some set of cubic identities.

In certain special cases we have been able to find an explicit formula for the general solution of the system (3). In the case in which \( p, r, \) and \( \lambda \) are equal to zero, the solution of the Cauchy problem with initial conditions \( u(0) = u_0 \) is given as

\[ u(t) = (E - tG)^{-1}(u_0), \]

where \( E \) is a unitary operator, \( G = L_{u^0_0} + [L_{u_0^0}, L_{u_0^0}] \). For any element \( x \) of the algebra \( \mathfrak{U} \), we denote by \( L_x \) the operator of left multiplication by \( x \) in \( \mathfrak{U} \). It is clear that \( u(t) \) does not have singularities for small \( t \).
Let us present the general solution for the system (3) with \( r = 0^* \) under the assumption that the initial condition is that \( u_0 \) is an invertible element in \( \mathfrak{u} \). Recall that an element \( x \in \mathfrak{u} \) is said to be invertible if \( \det P_x \neq 0 \), where \( P_x = 2(L_x)^2 - L(x^2) \). It may be verified that the function

\[
u = P^{-1}_r z,
\]

where

\[
z = \exp (\lambda t - L_p t) \left( P^{-1}_0 u_0 - \int_0^t \exp (L_p s - \lambda s)(\alpha) \, ds \right)
\]

is the solution of the Cauchy problem with invertible initial condition.

For inhomogeneous systems (3) with nonzero free term \( r \), we do not know of an explicit formula for the solution. However, the integrability by quadrature of these systems "almost" follows from a generalization of the Lie—Liouville theorem presented below.

**THEOREM 1.** Suppose that the vector field

\[
X = \sum_{i=1}^n f_i (x_1, \ldots, x_n) \frac{\partial}{\partial x_i},
\]

corresponding to the system of ordinary differential equations

\[
(x_i)_t = f_i (x_1, \ldots, x_n),
\]

is an element of a solvable Lie algebra \( \mathfrak{g} \) of vector fields whose basis \( Y_1, \ldots, Y_m \) we know explicitly. Then, if \( \mathfrak{g} \) is transitive (i.e., a matrix formed from the coefficients of the vector fields \( Y_1, \ldots, Y_m \) has rank \( n \) at a point of general position), the system (5) is integrable by quadrature.

**Proof.** It is known [2] that the basis \( Y_1, \ldots, Y_m \) in a solvable Lie algebra \( \mathfrak{g} \) may be chosen in such a way that \( I_k = \{ Y_1, \ldots, Y_m \} \) are ideals in \( \mathfrak{g} \) for any \( k = 1, 2, \ldots, m \). Suppose that \( i \) is a number such that the rank of the ideal \( I_i = \{ Y_1, \ldots, Y_i \} \) is equal to \( i \), and that the ideal \( I_{i-1} \) has rank \( i - 1 \). By the Frobenius theorem [3] there exists a nonconstant solution \( Y_1 (x_1, \ldots, x_n) \) of the system of linear equations \( Y_1 (y_1) = \ldots = Y_{i-1} (y_1) = 0 \). This solution is unique to within substitutions \( y_1 \rightarrow f(y_1) \). Inasmuch as \( [Y''_{i-1}, I_{i-1}] \subset I_{i-1} \), we have \( Y_{i-1} (y_1) = g(y_1) \) for some nonzero function \( g \). Without loss of generality, it may be assumed that \( g(y_1) = 1 \). Let us consider the set of equations

\[
Y_i (y) = 0, \ldots, Y_{i+i} (y) = 0, \ Y_n (y) = 1
\]

as a system of linear algebraic equations in the partial derivatives \( \partial y_i / \partial x_i, i = 1, \ldots, n \). It is clear that these partial derivatives may be uniquely determined from the system, after which the function \( y_1 \) is itself found by means of quadratures. After substituting for the variables \( x_1, \ldots, x_n \), we will assume that \( y_1 = x_1 \). Next, the coefficients of \( \partial / \partial x_1 \) in all the vector fields in \( I_{i-1} \) vanish, and having reduced the value of \( n \), we may apply the technique described above to the algebra \( I_{i-1} \). The fact that the coefficients of the vector fields belonging to \( I_{i-1} \) depend on \( x_1 \) as on parameters does not complicate the situation in the least. Continuing on with the procedure, we make use of quadratures, reducing the matrix consisting of the coefficients of the vector fields \( Y_1, \ldots, Y_m \) to stepwise form. For the field \( Y_{i-1} \) the coefficients of \( \partial / \partial x_{n+1} \) are equal to 1. It may be verified that as a result all the vector fields in \( \mathfrak{g} \) may be simultaneously reduced to the form

\[
Y = \sum_{i=1}^n (c_i x_i + f_i (x_1, \ldots, x_m)) \frac{\partial}{\partial x_i}
\]

The system of ordinary differential equations corresponding to a vector field of the form (6) is integrable by quadrature as a consequence of the fact that it is triangular.

**Remark.** In the case in which a solvable transitive Lie algebra forms a vector field (4) together with its symmetries, the integrability by quadrature of the system (5) was proved, it would appear, by Sophus Lie (cf. [4]). We have not encountered the theorem presented above in the literature. Despite its simplicity, it substantially enlarges the range of applicability of the symmetric Lie approach to the integrability of ordinary differential equations.

Let us explain why the Jordan wheels (3) are an ideal field for applications of Theorem 1.

*Quadratic systems with zero free term are often encountered in chemical kinetics.*