ON THE SPEED OF STABILIZATION OF THE 
SOLUTION OF A BOUNDARY PROBLEM FOR 
A PARABOLIC EQUATION

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Let $\Omega$ be an unbounded connected domain of the $n$-dimensional space $\mathbb{R}^n$, let $x = (x_1, x_2, \ldots, x_n)$ be a point of this space, and let $\Gamma$ be the boundary of the domain $\Omega$, where $\Gamma$ consists of a finite number of twice, continuously differentiable, connected, $(n-1)$-dimensional surfaces $\Gamma_i$, which have no points in common. The domain $\Omega$ lies to one side of each surface $\Gamma_i$.

Let us consider, in the cylinder $D = \Omega \times (t > 0)$, the parabolic equation

$$ \frac{\partial u}{\partial t} - L(u) \quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ii} \frac{\partial u}{\partial x_i} \right) - \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} - c(t, x) \cdot u(t, x) = 0, \quad (1) $$

whose coefficients we shall consider to be defined over the whole space $\mathbb{R}^n \times (t > 0)$, and to satisfy the following conditions:

1) the coefficients $a_{ij}(t, x)$ are elements of a real symmetric matrix $A(t, x)$, are continuous, bounded, and satisfy a Holder condition with respect to the variables $x$ and $t$; in addition, the derivatives $\partial a_{ij}(t, x) / \partial x_i$ ($i = 1, 2, \ldots, n$) exist, are continuous, bounded, and satisfy a Holder condition with respect to the variables $x$;

2) for an arbitrary vector $V$, the following inequality holds:

$$ 0 < \gamma |V|^2 \leq \Gamma \cdot A(t, x) \cdot V \leq \gamma |V|^2. \quad (2) $$

3) the coefficients $b_i(t, x)$ and $c(t, x)$ are continuous, bounded, satisfy a Holder condition with respect to the variables $x$, and also satisfy the inequalities

$$ c(t, x) \leq 0, \quad \sum_{i=1}^{n} \frac{\partial b_i(t, x)}{\partial x_i} \geq c(t, x), \quad (3) $$

$$ (b(t, x), n(x))_{\Gamma_t} = 0, \quad (4) $$

where $b(t, x)$ is a vector with coordinates $b_1(t, x), \ldots, b_n(t, x)$.

We shall be interested in the behavior, as $t \to \infty$, of the solution of Eq. (1), which, on $\Gamma$, satisfies the condition

$$ \frac{\partial u(t, x)}{\partial N} + z(t, x) \cdot u(t, x)_{\Gamma_t} = 0. \quad (5) $$

and the initial condition

$$ u(0, x) = \eta(x), \quad (6) $$

where

$$ \frac{\partial u(t, x)}{\partial N}_{\Gamma_t} = \sum_{i=1}^{n} a_{ii}(t, x) \xi_i \frac{\partial u(t, x)}{\partial x_i}_{\Gamma_t} $$

is the derivative with respect to the conormal, $n = \{\xi_1, \xi_2, \ldots, \xi_n\}$ is the vector of the exterior (with respect to $\Omega$) normal to the boundary $\Gamma$, and $g(t, x)$ is a function which is nonnegative, continuous, and bounded.

The solution is taken from a uniqueness class. Moreover, we note that the boundedness of the solution follows immediately from the boundedness of \( u(x) \) and the maximum principle.

The fundamental result of the present paper is the following:

**Theorem 1.** If the surface \( \Gamma \) is bounded, and if the space is of dimension \( n > 2 \), then the solution \( u(t, x) \) of the problem (1), (5), (6) satisfies, uniformly with respect to \( x \in \Omega \), the inequality

\[
|u(t, x)| < \frac{C}{t^{\frac{n}{2}}},
\]

where \( C \) is a constant.

Theorem 1 follows from the more general Theorem 2, in which boundedness of the surface \( \Gamma \) and the condition on the dimensionality of the space is not required, but, instead, the following a priori assumptions relative to the solution of the problem (1), (5), (6) are made.

A. For fixed \( t > 0 \), we have the relations

\[
\lim_{R \to \infty} \int_{S_R \cap \Omega} |\nabla u(t, x)| dS = 0,
\]

\[
\lim_{R \to \infty} \int_{S_R \cap \Omega} |u(t, x)| dS = 0,
\]

where \( S_R \) is a sphere of radius \( R \).

B. For an arbitrary integer \( k \geq 1 \), the function \( (u(t, x))^k \) may be continued onto the whole halfspace \( R^n \times (t \geq 0) \), so that the continuation \( \tilde{u}_k(t, x) \) that is obtained, together with its first derivatives with respect to the variables \( x \), is continuous over the whole half-space, the limit of the function \( \tilde{u}_k(t, x) \) exists as \( |x| \to \infty \) and is equal to zero, and the function \( \tilde{u}_k(t, x) \) satisfies the following inequalities:

\[
\sum_{V \in \Gamma} (\nabla \tilde{u}_k(t, x))^2 dx \leq C_1 \sum_{V \in \Gamma} (\nabla (u(t, x)))^2 dx,
\]

\[
\sum_{V \in \Gamma} |\tilde{u}_k(t, x)| dx \leq C_2 \sum_{V \in \Gamma} |u(t, x)|^2 dx,
\]

where the constants \( C_1 \) and \( C_2 \) depend only on the surface \( \Gamma \), the coefficients of the equation, and the dimensionality of the space.

**Theorem 2.** If conditions A and B are satisfied, the inequality (7) holds for the solution of the problem (1), (5), (6), uniformly with respect to \( x \in \Omega \).

1. We shall show, relying on the following lemma, whose proof will be given in subsection 3, that Theorem 1 follows from Theorem 2.

**Lemma 1.** Let the surface \( \Gamma \) be bounded. Then condition A is satisfied by the solution of the problem (1), (5), (6).

**Lemma 2.** Let the surface \( \Gamma \) be bounded, \( n > 2 \), and let \( g(t, x) \equiv 0 \). Then condition B is satisfied by the solution of the problem (1), (5), (6).

It follows, from the lemmas and from Theorem 2, that the inequality (7) holds in the case of the boundary condition

\[
\frac{\partial u(t, x)}{\partial N} |_{\Gamma} = 0.
\]

However, since \( g(t, x) \equiv 0 \), the modulus of the solution of the problem (1), (5), (6) can be majorized by that of the solution of the problem (1), (12), (6).