A Yang–Mills–Higgs Monopole of Charge 2

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Abstract. A new static, purely magnetic Yang–Mills–Higgs monopole solution is presented. It is axisymmetric and has a topological charge of 2; the charge is located at a single point.

1. Introduction

This paper is concerned with Yang–Mills–Higgs monopoles which are static and purely magnetic, in the Prasad–Sommerfield limit [1]. This means that we have a Higgs field $\phi$ and a gauge potential $A_j (j = 1, 2, 3)$ on Euclidean 3-space $\mathbb{R}^3$, satisfying the following five requirements.

(i) $\phi^a A^a_j$ take values in the Lie algebra of SU(2). In other words, $\phi$ and $A_j$ have the form $\phi = \phi^a \sigma^a$, $A_j = A^a_j \sigma^a$, where $\sigma^a$ are the Pauli matrices, and where $\phi^a, A^a_j$ are scalar functions on $\mathbb{R}^3$ which in some gauge are real-valued. (We shall be allowing SL(2, $\mathbb{C}$)-valued gauge transformations, so $\phi^a, A^a_j$ will not be real-valued in every gauge.)

(ii) In some gauge, $\phi$ & $A_j$ are smooth (say $C^\infty$) on $\mathbb{R}^3$.

(iii) The Bogomolny equations

$$G_{jk}^a = - \epsilon_{jkl} D^l \phi^a$$

are satisfied, where

$$G_{jk}^a = \partial_j A_k^a - \partial_k A_j^a + \kappa \epsilon^{abc} A_j^b A_k^c,$$

$$D_j \phi^a = \partial_j \phi^a + \kappa \epsilon^{abc} A_j^b \phi^c,$$

$\kappa$ being some real number (the coupling constant).

(iv) The norm $\| \phi \| = (\phi^a \phi^a)^{1/2}$ of the Higgs field has the asymptotic behaviour

$$\| \phi \| = 1 - m/r + O(r^{-2}) \text{ as } r \to \infty,$$

where $r$ is the Euclidean distance from the origin in $\mathbb{R}^3$, and $m$ is some real number.

(v) The energy

$$E = \int \left( \frac{1}{4} |G|^2 + \frac{1}{2} |D\phi|^2 \right) d^3 x$$

is finite. Here $|G|^2 = G_{jk}^a G_j^a$ and $|D\phi|^2 = (D_j \phi^a)(D_j \phi^a)$. 

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Remarks

(a) $\| \phi \|$, $\| G \|$ and $\| D\phi \|$ are invariant under SL(2, C) gauge transformations.

(b) Requirements (i)-(iv) in fact imply requirement (v). To see this, first note that the Bogomolny equation enables one to rewrite $E$ as

$$ E = \int \| D\phi \|^2 d^3 x. $$

From the Bogomolny equation and the Bianchi identities $D[I^I G_{k\ell}] = 0$ (square brackets denote skew-symmetrization) we get $D_j D_j \phi = 0$, and from this it follows that $\| D\phi \|^2 = \frac{1}{4} \hat{\partial}_j \hat{\partial}_j \| \phi \|^2$. Now letting $B_R$ denote the three-dimensional closed ball with radius $R$ in $\mathbb{R}^3$, we can write

$$ E = \lim_{R \to \infty} \int_{B_R} \frac{1}{2} \hat{\partial}_j \hat{\partial}_j \| \phi \|^2 d^3 x $$

$$ = \lim_{R \to \infty} \int_{B_R} \frac{1}{2} \hat{\partial}_j \| \phi \|^2 d^2 S j \text{ (Stokes’ theorem)} $$

$$ = 4\pi m $$

from requirement (iv). (The author is indebted to L. O’Raifeartaigh for supplying this argument.)

(c) The topological charge $n$ is defined by [2]

$$ n = \lim_{R \to \infty} \frac{1}{8\pi} \int_{\partial B_R} \xi_j d^2 S_j, $$

where $\xi_j = \epsilon_{jkl} \partial^a \hat{\phi}^b \hat{\phi}^c \hat{\phi}^d \hat{\phi}^e$ and $\hat{\phi}^a = \| \phi \|^{-1} \phi^a$. The number $n$ is necessarily an integer [2]. Its value is unchanged if we replace $\xi_j$ by $-\kappa \epsilon_{jkl} F_{k\ell}$, where $F_{k\ell}$ is the ’t Hooft magnetic tensor (see [2] Eq. (1), but beware the sign error contained therein). Now using the standard expression for $F_{jk}$ ([2], Eq. (1a)), the bound $\| D\phi \| = O(r^{-2})$ which follows from finiteness of energy, the Bogomolny equations, and the asymptotic form of $\| \phi \|$, one easily deduces that

$$ n = m\kappa. $$

The magnetic charge is defined to be $n/k$, and is therefore equal to $m$.

Up to now only one monopole solution was known: the spherically-symmetric Bogomolny-Prasad-Sommerfield (BPS) monopole, which has $n = 1$ [1]. One line of attack on the problem of finding further solutions was provided by the realization that the Bogomolny equations are equivalent to the self-dual Yang-Mills equations in Euclidean 4-space with the added condition that everything be independent of imaginary time; [3]. Indeed, the equations

$$ \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}^a = G_{\mu\nu}^a, \hat{\partial}_\nu A_\mu^a = 0 $$

are equivalent to

$$ G_{jk}^a = -\epsilon_{jkl} D_\ell A_\ell^a, \hat{\partial}_\nu A_\mu^a = 0, $$

and these are exactly the Bogomolny equations in 3-space, if we interpret $A_\mu$ as the Higgs field $\phi$. Manton [3] recognized that the BPS solution can be obtained out of the well-known “’t Hooft ansatz”, which expresses $A_\mu$ as functional of a