A Yang–Mills–Higgs Monopole of Charge 2

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Abstract. A new static, purely magnetic Yang–Mills–Higgs monopole solution is presented. It is axisymmetric and has a topological charge of 2; the charge is located at a single point.

1. Introduction

This paper is concerned with Yang–Mills–Higgs monopoles which are static and purely magnetic, in the Prasad–Sommerfield limit [1]. This means that we have a Higgs field $\phi$ and a gauge potential $A_j$ ($j = 1, 2, 3$) on Euclidean 3-space $\mathbb{R}^3$, satisfying the following five requirements.

(i) $\phi$ and $A_j$ take values in the Lie algebra of $SU(2)$. In other words, $\phi$ and $A_j$ have the form $\phi = \phi^a \sigma^a$, $A_j = A_j^a \sigma^a$, where $\sigma^a$ are the Pauli matrices, and where $\phi^a$, $A_j^a$ are scalar functions on $\mathbb{R}^3$ which in some gauge are real-valued. (We shall be allowing $SL(2, \mathbb{C})$-valued gauge transformations, so $\phi^a$, $A_j^a$ will not be real-valued in every gauge.)

(ii) In some gauge, $\phi$ and $A_j$ are smooth (say $C^\infty$) on $\mathbb{R}^3$.

(iii) The Bogomolny equations

$$G^a_{jk} = - \varepsilon_{jkl} D_k \phi^a$$

are satisfied, where

$$G^a_{jk} = \partial_j A_k^a - \partial_k A_j^a + \kappa \epsilon^{abc} A_j^b A_k^c,$$

$$D_k \phi^a = \partial_k \phi^a + \kappa \epsilon^{abc} A_j^b \phi^c,$$

$\kappa$ being some real number (the coupling constant).

(iv) The norm $\| \phi \| = (\phi^a \phi^a)^{1/2}$ of the Higgs field has the asymptotic behaviour

$$\| \phi \| = 1 - m/r + O(r^{-2}) \text{ as } r \to \infty,$$

where $r$ is the Euclidean distance from the origin in $\mathbb{R}^3$, and $m$ is some real number.

(v) The energy

$$E = \int (\frac{1}{4} \| G \|^2 + \frac{1}{2} \| D \phi \|^2) d^3x$$

is finite. Here $\| G \|^2 = G^a_{jk} G^a_{jk}$ and $\| D \phi \|^2 = (D_j \phi^a)(D_j \phi^a)$. 


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Remarks

(a) $\|\phi\|$, $\|G\|$ and $\|D\phi\|$ are invariant under $SL(2, \mathbb{C})$ gauge transformations.
(b) Requirements (i)-(iv) in fact imply requirement (v). To see this, first note that the Bogomolny equation enables one to rewrite $E$ as

$$E = \int \|D\phi\|^2 d^3x.$$ 

From the Bogomolny equation and the Bianchi identities $D_{(\mu}G_{\nu \lambda)} = 0$ (square brackets denote skew-symmetrization) we get $D_jD_j\phi = 0$, and from this it follows that $\|D\phi\|^2 = \frac{1}{2} \partial_j \partial_j \|\phi\|^2$. Now letting $B_R$ denote the three-dimensional closed ball with radius $R$ in $\mathbb{R}^3$, we can write

$$E = \lim_{R \to \infty} \int_{B_R} \frac{1}{2} \partial_j \partial_j \|\phi\|^2 d^3x = \lim_{R \to \infty} \int_{\partial B_R} \frac{1}{2} \partial_j \|\phi\|^2 d^2S^j \quad \text{(Stokes' theorem)}$$

from requirement (iv). (The author is indebted to L. O'Raifeartaigh for supplying this argument.)

(c) The topological charge $n$ is defined by [2]

$$n = \lim_{R \to \infty} \frac{1}{8\pi} \int_{\partial B_R} \xi_j d^2S^j,$$

where $\xi_j = \varepsilon_{jkl} \phi^{abc} \partial_k \partial_l \phi^a$ and $\phi^a = \|\phi\|^{-1} \phi^a$. The number $n$ is necessarily an integer [2]. Its value is unchanged if we replace $\xi_j$ by $-\kappa \varepsilon_{jkl} F_{kl}$, where $F_{kl}$ is the 't Hooft magnetic tensor (see [2] Eq. (1), but beware the sign error contained therein). Now using the standard expression for $F_{jk}$ ([2], Eq. (1a)), the bound $\|D\phi\| = O(r^{-2})$ which follows from finiteness of energy, the Bogomolny equations, and the asymptotic form of $\|\phi\|$, one easily deduces that

$$n = m \kappa.$$ 

The magnetic charge is defined to be $n/\kappa$, and is therefore equal to $m$.

Up to now only one monopole solution was known: the spherically-symmetric Bogomolny–Prasad–Sommerfield (BPS) monopole, which has $n = 1$ [1]. One line of attack on the problem of finding further solutions was provided by the realization that the Bogomolny equations are equivalent to the self-dual Yang–Mills equations in Euclidean 4-space with the added condition that everything be independent of imaginary time; [3]. Indeed, the equations

$$\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} G^a_{\mu \nu} = G^a_{\rho \sigma}, \quad \partial_0 A^a_\mu = 0$$

are equivalent to

$$G^a_{jk} = -\varepsilon_{jkl} D_l A^a_0, \quad \partial_0 A^a_\mu = 0;$$

and these are exactly the Bogomolny equations in 3-space, if we interpret $A_0$ as the Higgs field $\phi$. Manton [3] recognized that the BPS solution can be obtained out of the well-known "'t Hooft ansatz", which expresses $A_\mu$ as functional of a