This is a study concerning the buildup of an electron—photon avalanche in a medium with a random-distributed density. The formulas derived here describe the avalanche characteristics, averaged over fluctuations of the medium, in terms of the corresponding avalanche characteristics for a medium with a deterministically distributed density. Numerical results are shown pertaining to an avalanche in the atmosphere.

The probabilistic description of an electron—photon avalanche is a difficult mathematical problem. Such a description is usually rendered in either an A or a B approximation [1, 2]. It is even more difficult to analyze the effect which fluctuations of the properties of the medium have on the probabilistic characteristics of the avalanche. This problem arises, for instance, in the study of an avalanche buildup in a real atmosphere which is heterogeneous in space and variable in time. Another aspect of this problem has been touched upon by A. V. Stepanov [3]. It is the interpretation of a deterministic heterogeneous medium as a medium with random-distributed heterogeneities. A special case here is a medium with a periodic structure (an ionization calorimeter, for instance).

This study concerns the avalanche buildup in one direction only (along the z axis). The same model may be used also for the case of isotropic density fluctuations, if the correlation radius is larger than the transverse avalanche dimensions.

1. Equations of the Avalanche Buildup in a Heterogeneous Medium

We consider the avalanche buildup in a heterogeneous volume \( \mathcal{V} \) bounded by a nonconcave surface \( \mathcal{S} \). Let the sensitivity region of the detector be a flat surface \( \mathcal{S}^* \), and \( K_i(r^*, \omega; q) dq \) denote the probability that a detector reading \( q \in dq \) will result when particles of the \( i \)-th kind moving in the direction \( \omega \) with the energy \( E \) strike the surface \( \mathcal{S}^* \) at point \( r^* \). The corresponding probability for a particle located at point \( r \) will be denoted by \( \Psi_i(r, \omega; q) dq \).

Extending the results in [4], we apply the equation for \( \Psi_i \) to the given geometry

\[
- \omega \Psi_i = - \beta_i \frac{\partial \Psi_i}{\partial E} - \Sigma_i \Psi_i + \int dq' \int_0^\sigma dx [ \Sigma (i \rightarrow 1', 1'') \Psi_i \Psi_1' + \Sigma (i \rightarrow 1', 2'') \Psi_i \Psi_2' ] + \Psi_i
\]

with the boundary conditions

\[
\Psi_i(r, \omega, E, q) = \delta(q), \quad r \in \mathcal{S}, \quad \omega \cdot n > 0.
\]

Here \( i = 1 \) (electrons and positrons) or \( i = 2 \) (photons), \( \beta_i = \beta(r, E) \delta_{i1}, \Sigma (i \rightarrow 1', j'') \) is the differential cross section of the formation of \( i \)-particles moving in the \( \omega \)-direction with the energy \( E \) per couple of particles: namely of an electron with parameters \( \omega', E' \) and a \( j \)-particle with parameters \( \omega'', E'' \) at point \( r \) in the heterogeneous medium.


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\[ \Psi_i \equiv \Psi_i(r, \omega', E'; q'), \text{ } dx = d\omega'dE'd\omega''dE'' \text{, } \varphi_i = K_i(r^*, \omega, E; q), \]

\( S^{*\omega}_0(\mathbf{r}, \mathbf{S}^\omega) \) is the area of the projection of \( \mathbf{S}^\omega \) on a plane perpendicular to \( \omega \), \( n \) is the inward normal to \( \mathbf{S}_0 \) and \( q' + q'' = q \). The boundary conditions at the surface \( \mathbf{S}^\omega \) which separates, for instance, zones of volume \( V \) with various chemical compositions are defined, as conventionally, by the discontinuity of \( \Psi_i \) in the \( \omega \) direction [5].

From (1) and (2) follow the equations of moments

\[ M_{i,r} = \int_0^\infty q' \psi_i dq', \]

\[ -w \varphi M_{i,r} = -\beta_i \frac{\partial M_{i,r}}{\partial E} - \Sigma M_{i,r} + \sum_{k=0}^r \int dx \{ \Sigma (i \rightarrow 1', 1'') M_{i,k} + \Sigma (i \rightarrow 1', 2'') M_{i,2,k} \} M_{i,r-k} + m, \]

with the boundary conditions

\[ M_{i,r} (r, \omega, E) = \delta_i, \text{ } r \in S, \omega \cdot n > 0. \]

It is easy to ascertain that the equations of the first moment are coupled avalanche equations [6].

We now apply Eq. (1) to a plane-parallel layer the properties of which vary only along the \( z \) axis. Surface \( S \) splits here into two planes: \( z = z_0 \) and \( z^0 > z_0 \). Let \( \mathbf{S}^\omega \) also be an infinitely large plane \( z = z^* \) \((z_0 < z^* < z^0)\). In this case \( \Psi_i \) is not a function of \( x \) and \( y \), so that Eq. (1) becomes

\[ -\frac{\partial \Psi_i}{\partial z} = -\beta_i \frac{\partial \Psi_i}{\partial E} - \Sigma \psi_i \]

\[ + \int_0^q dq' \int dx \{ \Sigma (i \rightarrow 1', 1'') \Psi_i \psi_i + \Sigma (i \rightarrow 1', 2'') \Psi_i \psi_i + P \delta (z - z^*) \}, \]

(3)

where

\[ P = K_i(z^*, \omega, E; q), \]

and the boundary conditions are

\[ \Psi_i (z, \omega, E; q) = \delta (q), \text{ when } z = z_0 \text{ and } \omega z < 0, \text{ or } z = z^0 \text{ and } \omega z > 0. \]

4)

If \( \beta \) and \( \Sigma \) are not functions of \( z \), then (3) and (4) describe an avalanche process in a homogeneous layer. For a detector which counts the number of particles, \( q \) assumes discrete values and the integral with respect to \( q' \) in Eqs. (1) must be replaced by a sum, but function \( \psi \) must be integrated as a probability. With additional assumptions concerning the energy losses and the cross sections of interactions, Eq. (3) becomes the well-known equation of approximations A or B [4].

2. Similarity Transformation

We consider now the special case of a heterogeneous medium, namely a medium with a variable density but the same chemical composition throughout. Here the energy losses and the cross sections are functions of the coordinates, in terms of the density \( \rho(z) \):

\[ \beta (z, E) = \rho (z) \beta^m (E), \Sigma (z, E) = \rho (z) \Sigma^m (E), \]

while variables \( z \) and \( E \) in the coefficients of Eq. (3) are separable. Superscript \( m \) denotes the mass coefficients in a chemically homogeneous medium, independent of the coordinates. Equation (3) becomes

\[ -w \frac{\partial \Psi_i}{\partial z} = \rho (z) \left\{ -\frac{\partial \Psi_i}{\partial E} - \Sigma^m (E) \psi_i \right\} \]

\[ + \int_0^q dq' \int dx \{ \Sigma^m (i \rightarrow 1', 1'') \Psi_i \psi_i + \Sigma^m (i \rightarrow 1', 2'') \Psi_i \psi_i \} + P \delta (z - z^*). \]

(5)

In order to distinguish the solution to this equation from the solution for a layer with a uniform density, which will be used subsequently, we will place a tilde above the symbol \( \Psi_i \).

We now divide Eq. (5) by \( \rho (z) / \rho (\rho = \text{const} > 0) \) and change to a new unknown

\[ \tilde{z} (z) = \int_0^z \frac{\rho (z)}{\rho} dz. \]

(6)